University of Illinois at Urbana-Champaign

ECE 310: Digital Signal Processing

PROBLEM SET 2: SOLUTIONS

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Problem 1

i. $y[n-2] = x^2[n] + 3y[n]$
Non-linear, non-causal, shift-invariant.

ii. $y[n] = x[-n+2]$
Linear, non-causal, shift-varying.

iii. $y[n] = \left(\frac{1}{4}\right)^{|n|} x[n]$
Linear, causal, shift-varying.

iv. $y[n] = \sum_{m=-\infty}^{n+1} x[m]$
Linear, non-causal, shift-invariant.

v. $y[n] = \frac{x[n]}{x[2]}$
Nonlinear, non-causal, shift-varying

vi. $y[n-1] = x[n-1] + tan(4)x[n] - cos(0.4\pi n)y[n]$
Linear, non-causal, shift-varying.

vii. $y[n] = x[12n]$
Linear, non-causal, shift-varying.

Problem 2

a) The inverse DFT of the sequence $X[k] = \{1, e^{-j\pi/2}, 0, e^{j\pi/2}\}$, where the first entry of $X[k]$ corresponds to $k = 0$ is given by

$$\begin{aligned}
x[n] &= \frac{1}{N}\sum_{k=0}^{N-1} e^{j\frac{2\pi nk}{N}} \\
&= \frac{1}{4}\left(X[0] + X[1]e^{j\frac{\pi}{2}} + X[2]e^{j\pi n} + X[3]e^{j\frac{3\pi}{2}}\right) \text{ for } n = 0, 1, 2, 3 \\
&= \frac{1}{4}\left(1 + 2cos\left(\frac{\pi}{2}(n-1)\right)\right)
\end{aligned}$$

Hence $x[0] = \frac{1}{4}, x[1] = \frac{3}{4}, x[2] = \frac{1}{4}, x[3] = \frac{-1}{4}$

b) Using the circular time shift property of the DFT

$$\begin{aligned}
Y[k] &= X[k]e^{\frac{-j\pi k}{2}} \\
y[n] &= \frac{1}{4}\sum_{k=0}^{3} Y[k]e^{\frac{2\pi nk}{4}} \\
&= \frac{1}{4}\sum_{k=0}^{3} X[k]e^{\frac{-j\pi k}{2}}e^{\frac{2\pi nk}{4}} \\
&= \frac{1}{4}\sum_{k=0}^{3} X[k]e^{\frac{2\pi(n-1)k}{4}} \\
&= x[< n-1 >_4]
\end{aligned}$$

Problem 3

a) Using circular time shift property

$$\begin{aligned} x_1[n] &= x[< n-5 >_8] \\ X_1[k] &= X[k]e^{\frac{-j2\pi 5k}{8}} \\ &= X[k]e^{\frac{-j5\pi k}{4}} \end{aligned}$$

b)

$$\begin{aligned} x_2[n] &= x[< n-3 >_8] \\ X_2[k] &= X[k]e^{\frac{-j2\pi 3k}{8}} \\ &= X[k]e^{\frac{-j3\pi k}{4}} \end{aligned}$$

Problem 4

a. $y_1[n] = x[n] * h[n]$, where $h[n] = (0.7)^n u[n]$

$$\begin{aligned} y_1[n] &= \sum_{k=-\infty}^{\infty} h[n-k]x[k] \\ &= \sum_{k=-\infty}^{\infty} (0.7)^{n-k}u[n-k](0.9)^k u[k] \\ &= (0.7)^n \sum_{k=0}^{n} (0.7)^{-k}(0.9)^k u[n] \\ &= (0.7)^n \frac{1-(0.9/0.7)^{n+1}}{1-0.9/0.7} u[n] \\ &= \frac{(0.7)^{n+1}-(0.9)^{n+1}}{0.7-0.9} u[n] \\ &= 5\left(0.9^{n+1}-0.7^{n+1}\right)u[n] \\ &= [4.5(0.9)^n - 3.5(0.7)^n]u[n] \end{aligned}$$

b. $y_2[n] = x[n] * w[n]$, where $w[n] = u[n] - u[n-10]$
Using graphical convolution, it is easy to see that there are three distinct cases:

- for $n < 0$

$$y_2[n] = 0$$

- for $0 \leq n \leq 9$

$$\begin{aligned} y_2[n] &= \sum_{k=0}^{n} (0.9)^k \\ &= \sum_{k=0}^{\infty} (0.9)^k - \sum_{k=n+1}^{\infty} (0.9)^k \\ &= \frac{1-0.9^{n+1}}{1-0.9} \\ &= 10(1-0.9^{n+1}) \end{aligned}$$

- for $n \geq 10$

$$\begin{aligned}
y_2[n] &= \sum_{k=n-9}^{n} (0.9)^k \\
&= \frac{(0.9)^{n-9} - (0.9)^{n+1}}{1 - 0.9} \\
&= 10(0.9)^{n-9}\left[1 - (0.9)^{10}\right]
\end{aligned}$$

c. $y[n] = x[n] * v[n]$, where $v[n] = \frac{1}{5}h[n-1] + \frac{1}{10}w[n+9]$

$$\begin{aligned}
y[n] &= x[n] * v[n] \\
&= \frac{1}{5}(x[n] * h[n-1]) + \frac{1}{10}(x[n] * w[n+9]) \\
&= \frac{1}{5}y_1[n-1] + \frac{1}{10}y_2[n+9]
\end{aligned}$$

Now from parts (a) and (b) we have,

$$\frac{1}{5}y_1[n-1] = \begin{cases} 0, & n < 1 \\ (0.9)^n - (0.7)^n, & n \geq 1 \end{cases}$$

$$\frac{1}{10}y_2[n+9] = \begin{cases} 0, & n < -9 \\ (1 - (0.9)^{n+10}), & -9 \leq n \leq 0 \\ (0.9)^n(1 - (0.9)^{10}), & n \geq 1 \end{cases}$$

Combining the above two results together we have,

$$y[n] = \begin{cases} 0, & n < -9 \\ (1 - (0.9)^{n+10}), & -9 \leq n \leq 0 \\ (0.9)^n(2 - (0.9)^{10}) - 0.7^n, & n \geq 1 \end{cases}$$

d. $y_0[n] = s[n] * t[n]$, where $s[n] = x[n-3]$ and $t[n] = h[n+1]$

$$\begin{aligned}
y_0[n] &= s[n] * h[n] \\
&= x[n-3] * h[n+1] \\
&= y_1[n-3+1] \\
&= y_1[n-2]
\end{aligned}$$

which yields,

$$y_0[n] = 5\left[(0.9)^{n-1} - (0.7)^{n-1}\right]u[n-2]$$

therefore,

$$y_0[n] = \begin{cases} 0, & n < 2 \\ 5\left[(0.9)^{n-1} - (0.7)^{n-1}\right], & n \geq 2 \end{cases}$$

Problem 5

For $x[n] = \delta[n]$ we have,

$$\begin{aligned}
y[n] - \frac{1}{4}y[n-1] &= 4x[n] + 3x[n-1] \\
y[n] - \frac{1}{4}y[n-1] &= 4\delta[n] + 3\delta[n-1]
\end{aligned}$$

Consider the following cases:

1. $n = 0$,

$$\begin{aligned}
h[0] - \frac{1}{4}h[-1] &= 4 \\
h[0] &= 4
\end{aligned}$$

2. $n = 1$,

$$h[1] - \frac{1}{4}h[0] = 3$$
$$h[1] = 4$$

3. $n \geq 2$

$$h[n] - \frac{1}{4}h[n-1] = 0$$
$$h[n] = \frac{1}{4}h[n-1]$$

Hence the solution is of the form,

$$h[n] = K\left(\frac{1}{4}\right)^n$$

We can find K by using $h[1]$

$$h[1] = K\left(\frac{1}{4}\right)^1 = 4$$
$$K = 16$$

Therefore,

$$h[n] = \begin{cases} 4, & n = 0 \\ 16\left(\frac{1}{4}\right)^n, & n \geq 1 \end{cases}$$

or

$$h[n] = -12\delta[n] + 16\left(\frac{1}{4}\right)^n u[n]$$

Problem 6

The system can be represented as,

$$y[n] - \frac{1}{4}y[n-1] = 4x[n] + 3x[n-1]$$

this is the same difference equation as in Problem 5. Therefore the system output is given by,

$$\begin{aligned}
y[n] &= h[n] * x[n] \\
&= \sum_{k=-\infty}^{\infty} (0.5)^k u[k-2]\left(-12\delta[n-k] + 16\left(\frac{1}{4}\right)^{n-k} u[n-k]\right) \\
&= -12\left((0.5)^n u[n-2]\right) + \sum_{k=-\infty}^{\infty} (0.5)^k u[k-2]\left[16\left(\frac{1}{4}\right)^{n-k} u[n-k]\right] \\
&= -12\left((0.5)^n u[n-2]\right) + 16\sum_{k=-\infty}^{\infty} (0.5)^k u[k-2]\left(\frac{1}{4}\right)^{n-k} u[n-k] \\
&= -12\left((0.5)^n u[n-2]\right) + 16\left(\frac{1}{4}\right)^n \sum_{k=-\infty}^{\infty} (0.5^k) u[k-2]\left(\frac{1}{4}\right)^{-k} u[n-k] \\
&= -12\left((0.5)^n u[n-2]\right) + 16\left(\frac{1}{4}\right)^n \sum_{k=2}^{\infty} 2^k u[n-k] \\
&= -12\left((0.5)^n u[n-2]\right) + 16\left(\frac{1}{4}\right)^n \sum_{k=2}^{n} 2^k u[n-2]
\end{aligned}$$

The sum $\sum_{k=2}^{n} 2^k$ can be written as,

$$\sum_{k=2}^{n} 2^k = \sum_{m=0}^{n-2} 2^{m+2} = 4\frac{1-2^{n-1}}{1-2} = 4(2^{n-1}-1)$$

Therefore,

$$\begin{aligned} y[n] &= -12\left((0.5)^n u[n-2]\right) + 64\left(\frac{1}{4}\right)^n \left(2^{n-1}-1\right) u[n-2] \\ &= \left(-12.(0.5)^n + 16.(0.5)^n - \left(\frac{1}{4}\right)^{n-3}\right) u[n-2] \\ &= \left(4.(0.5)^n - \left(\frac{1}{4}\right)^{n-3}\right) u[n-2] \end{aligned}$$

Problem 7

a. The DTFT is shown in Fig. 1 and Fig. 2 respectively,

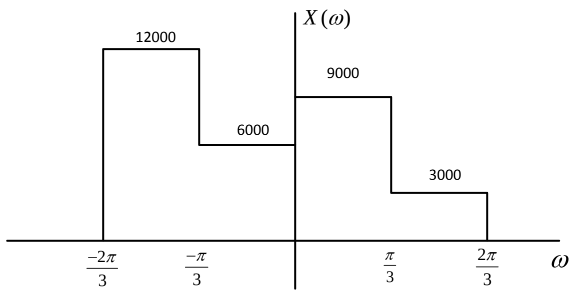

Figure 1: Problem 7.DTFT for Sampling Frequency $F_s = 3000$ Hz

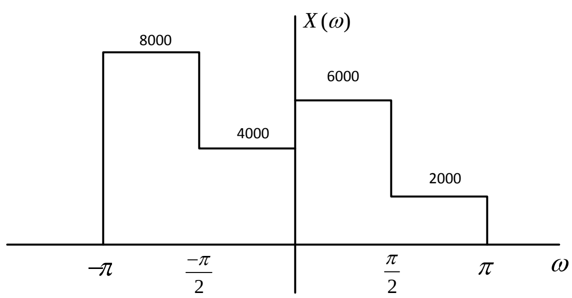

Figure 2: Problem 7.DTFT for Sampling Frequency $F_s = 2000$ Hz

b. The minimum sampling rate to avoid aliasing is given by the Nyquist rate $f_s \geq \frac{1}{T_{max}}$. For our given signal, this occurs at the sampling rate $T = 1000$ seconds. This Nyquist rate is 2000 Hz.

Problem 8

a. The highest frequency in the signal is $F = 2Hz$. Therefore, the Nyquist frequency is $F_{Nyquist} = 4Hz$

b. —Let $X_d(\omega)$ denote the DTFT of the signal. The 512 point DFT is given by,

$$X[k] = X_d(\frac{2\pi k}{N})$$

where $N = 512$.
i.e ω is sampled at 512 equally spaced point,

$$\omega_k = \frac{2\pi k}{512}$$

which yields,

$$k = \frac{\omega_k N}{2\pi}$$

We will observe a peak at $\omega_k = \frac{4\pi}{10}$ (which corresponds to the discrete frequency of the term $cos(4\pi t)$) and its second copy at $\omega_k^{'} = 2\pi - \frac{4\pi}{10} = \frac{16\pi}{10}$ due to the symmetry and periodicity of the spectrum. Hence the largest amplitude of DFT are at,

$$k = \lfloor \frac{4\pi}{10(2\pi)} \times 512 \rfloor = 102$$

and

$$k^{'} = \lfloor \frac{16\pi}{10(2\pi)} \times 512 \rfloor = 409$$

c. We cannot use 50% overlap criteria as the small peak may completely merge into the larger. We will use the no overlap method,

$$N > \frac{4\pi}{(\Omega_1 - \Omega_0)T}$$
$$N > \frac{4\pi \times 10}{4\pi - \frac{63\pi}{16}}$$
$$= \frac{4\pi \times 10 \times 16}{\pi}$$

Therefore $N > 640$

d. Larger T will give better resolution, as all frequencies are farther apart. The Nyquist frequency will give the best resolution,

$$(\Omega_1 - \Omega_0) > \frac{4\pi \times 4}{640} = \frac{\pi}{40} \text{ rad/s}$$

Hence $f = \frac{1}{80}$ Hz.

e. As f_s is increased, all analog frequencies are squeezed in and therefore come closer to each other in the spectral analysis and this degrades the resolution capability of the spectral analysis method.

Problem 9

a. We must have $x[n] = x_a(nT)$ where T is the sampling period. From the given expressions of $x[n]$ and $x_a(nT)$ we have,

$$\frac{\pi n}{5} = 15\pi nT$$
$$\frac{6\pi n}{5} = 90\pi nT$$

Hence $T = \frac{1}{75}$.

b. The choice is not unique. We can have,

$$\frac{\pi n}{5} + 2\pi kn = 15\pi nT$$
$$\frac{6\pi n}{5} + 2\pi kn = 90\pi nT$$

Hence any solution of the form $T = \frac{1+10k}{75}, k = 0, 1, 2, \ldots$ will work.
Problem 10

a. The sampled signal is,

$$x[n] = x_a(nT) = cos(450\pi nT)$$

Let $X_c(\Omega)$ be the Fourier Transform of $x_a(t)$, then we have,

$$\begin{aligned} X_c(\Omega) &= \frac{1}{2} F\left[e^{j450\pi t} + e^{-j450\pi t}\right] \\ &= \frac{1}{2}\left[2\pi\delta(\Omega - 450\pi) + 2\pi\delta(\Omega + 450\pi)\right] \\ &= \pi\left[\delta(\Omega - 450\pi) + \delta(\Omega + 450\pi)\right] \end{aligned}$$

The Fourier Transform of $X_a(t)$ is shown in Fig. 3. The Nyquist sampling rate is $T_s = \frac{\pi}{\Omega_{max}} = \frac{\pi}{450\pi} = 2.2ms$. Now,

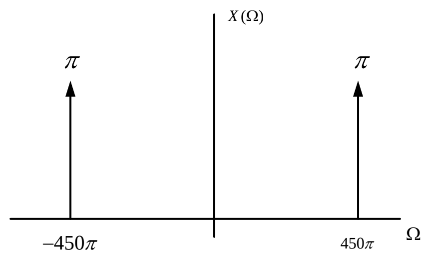

Figure 3: Problem 10. Fourier Transform of $x_a(t)$.

$T = 1ms < T_s$. By the sampling theorem we have,

$$X_d(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\left(\frac{\omega + 2n\pi}{T}\right)$$

where $\omega = \Omega T = 0.45\pi$. The DTFT of $x[n]$ for $T = 1ms$ is shown in Fig. 4. When $T = 2.5ms > T_s$ we will have aliasing.

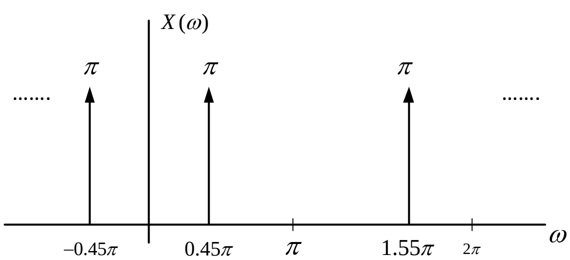

Figure 4: Problem 10. Fourier Transform of $x[n]$ for $T = 1$ ms.

$\omega = \Omega T = 450\pi \times 2.5(ms) = 1.125\pi$. The Fourier Transform is shown in Fig. 5.

b. Since the Nyquist sampling rate $T_s = \frac{1}{450}$s. The maximum sampling period $T_{max} = T_s = \frac{1}{450}$s, such that no aliasing occurs.

Problem 11 The difference equation is

$$y[n] = Ay[n-1] + By[n-2] \tag{1}$$

where $A = 1/3, B = -2/3$ In general solution of the form

$$y[n] = cr^n$$

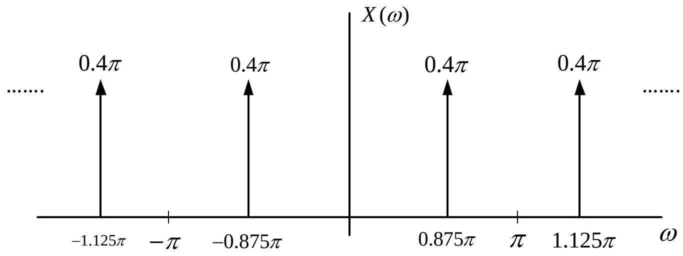

Figure 5: Problem 10. Fourier Transform of $x[n]$ for $T = 2.5$ ms.

Replacing $y[n]$ in (1) and simplifying:

$$
\begin{aligned}
cr^n &= Acr^{n-1} + Bcr^{n-2} \\
r^n &= Ar^{n-1} + Br^{n-2} \\
r^{n-2}r^2 &= r^{n-2}(Ar + B) \\
r^2 &= Ar + B
\end{aligned}
$$

which can be written as

$$r^2 - Ar - B = 0 \tag{2}$$

Solving for the roots of (2) we get the roots as $r_1 = \alpha + j\beta$ and $r_2 = \alpha - j\beta$, where $\alpha = \frac{-1}{6}$ and $\beta = \frac{\sqrt{23}}{6}$. $y[n]$ can now be written as,

$$
\begin{aligned}
y[n] &= cr_1^n + dr_2^n \\
&= c(\alpha + j\beta)^n + d(\alpha + j\beta)^n \\
&= \left(\sqrt{\alpha^2 + \beta^2}\right)^n \left(ce^{-j\theta n} + de^{j\theta n}\right) \\
&= 2\left(\sqrt{\alpha^2 + \beta^2}\right)^n (Ecos(\theta n) + Fsin(\theta n))
\end{aligned} \tag{3}
$$

where, $c = E + jF$, $d = E - jF$, $\theta = tan^{-1}\frac{\beta}{\alpha}$ Now substituting the values of α, β we get,

$$y[n] = 2\left(\frac{\sqrt{6}}{3}\right)^n (Ecos(78.22n) + Fsin(78.22n))$$

We can now use initial conditions, $y[-1]$ and $y[-2]$ to solve for E and F

$$
\begin{aligned}
1 &= 2\left(\frac{\sqrt{6}}{3}\right)(Ecos(78.22) + Fsin(78.22)) \\
-1 &= 2\left(\frac{\sqrt{6}}{3}\right)^2 (Ecos(78.22 * 2) + Fsin(78.22 * 2))
\end{aligned}
$$

Solving we get $E = 0.5$ and $F = -0.3127$ hence the solution is

$$y[n] = 2\left(\frac{\sqrt{6}}{3}\right)^n (0.5cos(78.22n) - 0.3127sin(78.22n))$$