University of Illinois at Urbana-Champaign

ECE 310: Digital Signal Processing

PROBLEM SET 3: SOLUTIONS

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Problem 1

a) $x[n] = \left(\frac{1}{3}\right)^n u[n]$

$$X(z) = \frac{1}{1 - \frac{1}{3}z^{-1}}, \quad ROC : |z| > \frac{1}{3}$$

b) $x[n] = \left(\frac{1}{3}\right)^n u[n-3]$

$$x[n] = \left(\frac{1}{3}\right)^3 \left(\frac{1}{3}\right)^{n-3} u[n-3]$$

Hence,

$$X(z) = \left(\frac{1}{3}\right)^3 z^{-3} \frac{1}{1 - \frac{1}{3}z^{-1}}, \quad ROC : |z| > \frac{1}{3}$$

c) $x[n] = n^2 u[n]$
Let $x_1[n] = nu[n]$, we then have

$$\begin{aligned} X_1(z) &= -z \frac{d}{dz} \frac{1}{1 - z^{-1}} \\ &= \frac{z^{-1}}{(1 - z^{-1})^2} \end{aligned}$$

Now $x[n] = nx_1[n]$, hence

$$X(z) = -z \frac{dX_1(z)}{dz}$$
$$X(z) = \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}$$

d) $x[n] = e^{j\pi n/3} u[n]$

$$x[n] = \left(e^{j\pi/3}\right)^n u[n]$$
$$\Rightarrow X(z) = \frac{1}{1 - e^{j\pi/3} z^{-1}}, \quad ROC : |z| > 1$$

e) $x[n] = sin(\omega n + \theta) u[n]$

$$x[n] = \frac{1}{2j} \left(e^{j\omega n} e^{j\theta} - e^{-j\omega n} e^{-j\theta}\right) u[n]$$
$$\begin{aligned} \Rightarrow X(z) &= \frac{1}{2j} \left(\frac{e^{j\theta}}{1 - e^{j\omega} z^{-1}} - \frac{e^{-j\theta}}{1 - e^{-j\omega} z^{-1}} \right) \\ &= \frac{sin\theta + sin(\omega - \theta) z^{-1}}{1 - 2cos(\omega) z^{-1} + z^{-2}}, \quad ROC : |z| > 1 \end{aligned}$$

f) $x[n] = n\left(\frac{1}{2}\right)^n u[n]$

$$\begin{aligned} X(z) &= -z\frac{d}{dz}\frac{1}{1-\frac{1}{2}z^{-1}} \\ &= \frac{\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})^2}, \quad ROC: |z| > \frac{1}{2} \end{aligned}$$

Problem 2

a)

$$\begin{aligned} \frac{X(z)}{z} &= \frac{z-1}{z^2+3z+2} \\ &= \frac{-2}{z+1} + \frac{3}{z+2} \\ X(z) &= \frac{-2}{1+z^{-1}} + \frac{3}{1+2z^{-1}} \\ x[n] &= -2(-1)^n u[n] + 3(-2)^n u[n] \end{aligned}$$

b)

$$\begin{aligned} X(z) &= \frac{\frac{1}{j\sqrt{2}}e^{j\pi/4}}{1+\frac{1}{\sqrt{2}}e^{j\pi/4}z^{-1}} + \frac{\frac{-1}{j\sqrt{2}}e^{-j\pi/4}}{1+\frac{1}{\sqrt{2}}e^{-j\pi/4}z^{-1}} \\ x[n] &= \left[\frac{1}{j\sqrt{2}}e^{j\pi/4}\left(\frac{-1}{\sqrt{2}}e^{j\pi/4}\right)^n - \frac{1}{j\sqrt{2}}e^{-j\pi/4}\left(\frac{-1}{\sqrt{2}}e^{-j\pi/4}\right)^n\right]u[n] \\ &= -\frac{1}{j}\left[\left(-\frac{1}{\sqrt{2}}e^{j\pi/4}\right)^{n+1} - \left(-\frac{1}{\sqrt{2}}e^{-j\pi/4}\right)^{n+1}\right]u[n] \\ &= -2\left(-\frac{1}{\sqrt{2}}\right)^{n+1} sin\left(\frac{\pi(n+1)}{4}\right)u[n] \\ &= (-1)^n\left(\frac{1}{\sqrt{2}}\right)^{n-1} sin\left(\frac{\pi(n+1)}{4}\right)u[n] \end{aligned}$$

Problem 3

a) z-transform does not exist. A valid ROC cannot be found, $|z| > 1/2$ and $|z| < 1/2$

b)

$$\begin{aligned} x[n] &= \left(\frac{1}{2}\right)^{|n|} \\ &= \left(\frac{1}{2}\right)^{-n} u[-n-1] + \left(\frac{1}{2}\right)^n u[n] \\ X(z) &= \frac{-1}{1-2z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \frac{1}{2} < |z| < 2 \end{aligned}$$

c) $x[n] = \delta[n] \Rightarrow X(z) = 1$, ROC: entire z-plane d)

$$\begin{aligned}
x[n] &= \left(\frac{1}{3}\right)^n u[-n+1] \\
&= \frac{1}{9}\left(\frac{1}{3}\right)^{n-2} u[-(n-2)-1] \\
&= \frac{1}{9} x_1[n-2] \\
\text{where, } x_1[n] &= -\left(\frac{1}{3}\right)^n u[-n-1] \\
\Rightarrow X(z) &= -\frac{1}{9} z^{-2} \frac{1}{1-\frac{1}{3}z^{-1}}, |z| < 1/3
\end{aligned}$$

e)

$$\begin{aligned}
x[n] &= nu[n] - (n-N)u[n-N] \\
\Rightarrow X(z) &= -z\frac{d}{dz}\frac{1}{1-z^{-1}} - z^{-N}\left(-z\frac{d}{dz}\frac{1}{1-z^{-1}}\right) \\
&= \frac{z^{-1}(1-z^{-N})}{(1-z^{-1})^2}, \quad |z| > 1
\end{aligned}$$

Problem 4

a)

$$\begin{aligned}
X_1(z) &= \frac{z}{z-a}, \quad |z| > |a| \\
X_2(z) &= \frac{b^3 z^{-2}}{z-b}, \quad |z| > |b| \\
Y(z) &= \frac{z^3(z-b) + b^3(z-a)}{z^2(z-a)(z-b)}, \quad |z| > |a|
\end{aligned}$$

b)

$$\begin{aligned}
X(z) &= 2 - z^{-3}, \quad |z| > 0 \\
Y(z) &= H(z)X(z) = \frac{2z^3 - 1}{z^3(z+4)(z+2)}
\end{aligned}$$

Possible ROC's of $Y(z)$ are
Case I : $|z| > 4$
Case II : $4 > |z| > 2$
Case III : $2 > |z| > 0$

Problem 5

The difference equation of the system is,

$$\begin{aligned}
y[n] - ay[n-1] &= x[n] \\
\Rightarrow Y(z) - az^{-1}(Y(z) + zy[-1]) &= X(z) \\
Y(z) &= \frac{1}{1-az^{-1}}\left(X(z) + ay[-1]\right)
\end{aligned}$$

$$y[-1] = 1 \text{ and } x[n] = b^n u[n] \Rightarrow X(z) = \frac{1}{1-bz^{-1}}, |z| > |b|$$

$$\Rightarrow Y(z) = \frac{1}{(1-az^{-1})(1-bz^{-1})} + \frac{a}{1-az^{-1}}$$
$$= \frac{\frac{a}{a-b}+a}{(1-az^{-1})} - \frac{\frac{b}{a-b}+a}{(1-bz^{-1})}, \quad |z| > |b| > |a|$$
$$y[n] = \left(\frac{a^{n+1}-b^{n+1}}{a-b} + a^{n+1}\right) u[n]$$

Problem 6

a)

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$
$$= \sum_{m=-\infty}^{n-11} x[m]\left(\frac{1}{3}\right)^{n-m} + \sum_{m=n-10}^{n+1} x[m]$$

Consider now the following 3 cases:
Case I: For $n+1 \leq 2 \Rightarrow n \leq 1$

$$y[n] = \sum_{m=-\infty}^{n-11} 2^m \left(\frac{1}{3}\right)^{n-m} + \sum_{m=n-10}^{n+1} 2^m$$
$$= 3^{-n} \sum_{m=-\infty}^{n-11} 6^m + \sum_{n-10}^{n+1} 2^m$$
$$= 3^{-n}\left(\frac{6^{n-10}}{5}\right) + \left(\frac{2^{n-10}-2^{n+2}}{-1}\right)$$
$$= \left(\frac{1}{5}\right) 3^{-n}6^n6^{-10} + 2^n2^2 - 2^n2^{-10}$$
$$= \left[4 + \frac{1}{5}\left(\frac{1}{6}\right)^{10} - 2^{-10}\right] 2^n$$

Case II : For $n+1 > 2$ and $n-11 \leq 2 \Rightarrow 2 \leq n \leq 13$

$$y[n] = \sum_{m=-\infty}^{n-11} 2^m \left(\frac{1}{3}\right)^{n-m} + \sum_{m=n-10}^{2} 2^m + \sum_{m=3}^{n+1}\left(\frac{1}{2}\right)^m$$
$$= \frac{6^{-10}}{5}2^n + \left(\frac{2^{n-10}-2^3}{-1}\right) + \left(\frac{\left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^{n+2}}{\frac{1}{2}}\right)$$
$$= \left(\frac{1}{5}\left(\frac{1}{6}\right)^{10} - 2^{-10}\right)2^n - \left(\frac{1}{2}\right)2^{-n} + \frac{33}{4}$$

Case III : For $n-11>2 \Rightarrow n>13$

$$\begin{aligned}
y[n] &= \sum_{m=-\infty}^{2} 2^m \left(\frac{1}{3}\right)^{n-m} + \sum_{m=3}^{n-11} \left(\frac{1}{2}\right)^m \left(\frac{1}{3}\right)^{n-m} + \sum_{m=n-10}^{n+1} \left(\frac{1}{2}\right)^m \\
&= 3^{-n} \sum_{m=-\infty}^{2} 6^m + 3^{-n} \sum_{m=3}^{n-11} \left(\frac{3}{2}\right)^m + \sum_{n-10}^{n+1} \left(\frac{1}{2}\right)^m \\
&= 3^{-n} \left(\frac{6^3}{5}\right) + 3^{-n} \left(\frac{\left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right)^{n-10}}{\frac{-1}{2}}\right) + \left(\frac{\left(\frac{1}{2}\right)^{n-10} - \left(\frac{1}{2}\right)^{n+2}}{\frac{1}{2}}\right) \\
&= \frac{6^3}{5} 3^{-n} - \frac{\left(\frac{3}{2}\right)^3}{\frac{1}{2}} 3^{-n} + \frac{\left(\frac{3}{2}\right)^{-10}}{\frac{1}{2}} 2^{-n} + \left(\frac{1}{2}\right)^{-11} 2^{-n} - \frac{1}{2} 2^{-n} \\
&= \left[\left(\frac{1}{2}\right)^{-11} \left(\frac{1}{3}\right)^{10} + \left(\frac{1}{2}\right)^{-11} - \frac{1}{2}\right] 2^{-n} + \frac{729}{10} \left(\frac{1}{3}\right)^n
\end{aligned}$$

b.

$$
\begin{aligned}
H(z) &= \frac{z^2}{z-1} + \left(\frac{1}{3}\right)^{11} \frac{z^{-10}}{z-\frac{1}{3}} - \frac{z^{-10}}{z-1} \\
&= \frac{z^2 - z^{-10}}{z-1} + \left(\frac{1}{3}\right)^{11} \frac{z^{-10}}{z-\frac{1}{3}}, \quad |z| > 1 \\
x[n] &= 2^n u[-n+2] + \left(\frac{1}{2}\right)^n u[n-3] \\
&= 8(2)^{n-3} u[-(n-3)-1] + \frac{1}{8}\left(\frac{1}{2}\right)^{n-3} u[n-3] \\
X(z) &= -8\frac{z^{-2}}{z-2} + \frac{1}{8}\frac{z^{-2}}{z-\frac{1}{2}}, \quad 2 > |z| > \frac{1}{2} \\
Y(z) &= H(z)X(z) \\
&= \frac{1}{8}\frac{1-z^{-12}}{(z-1)(z-\frac{1}{2})} - 8\frac{1-z^{-12}}{(z-1)(z-2)} + \frac{1}{8}\left(\frac{1}{3}\right)^{11}\frac{z^{-12}}{(z-\frac{1}{3})(z-\frac{1}{2})} - 8\left(\frac{1}{3}\right)^{11}\frac{1-z^{-12}}{(z-\frac{1}{3})(z-2)}, \quad 2 > |z| > 1 \\
&= \frac{1}{8}(z^{-1} - z^{-13})\left(\frac{z}{(z-1)(z-\frac{1}{2})}\right) - 8(z^{-1} - z^{-13})\left(\frac{z}{(z-2)(z-1)}\right) + \frac{1}{8}\left(\frac{1}{3}\right)^{11} z^{-13}\left(\frac{z}{(z-\frac{1}{3})(z-\frac{1}{2})}\right) \\
&- 8\left(\frac{1}{3}\right)^{11} z^{-13}\left(\frac{z}{(z-\frac{1}{3})(z-2)}\right) \\
&= \frac{1}{8}(z^{-1} - z^{-13})\left(\frac{2}{z-1} - \frac{1}{z-\frac{1}{2}}\right) - 8(z^{-1} - z^{-13})\left(\frac{2}{(z-2)} - \frac{1}{(z-1)}\right) \\
&+ \frac{1}{8}\left(\frac{1}{3}\right)^{11} z^{-13}\left(\frac{3}{z-\frac{1}{2}} - \frac{2}{z-\frac{1}{3}}\right) - 8\left(\frac{1}{3}\right)^{11} z^{-13}\left(\frac{\frac{6}{5}}{(z-2)} - \frac{\frac{1}{5}}{(z-\frac{1}{3})}\right) \\
&= \frac{33}{4}z^{-2}\frac{z}{z-1} - \frac{33}{4}z^{-14}\frac{z}{z-1} - 16z^{-2}\frac{z}{z-2} + 16\left(1 - \frac{1}{5}\left(\frac{1}{3}\right)^{10}\right) z^{-14}\frac{z}{z-2} \\
&- \frac{1}{8}z^{-2}\frac{z}{z-\frac{1}{2}} + \frac{1}{8}\left(\left(\frac{1}{3}\right)^{10} + 1\right) z^{-14}\frac{z}{z-\frac{1}{2}} + \frac{27}{20}\left(\frac{1}{3}\right)^{11} z^{-14}\frac{z}{z-\frac{1}{3}}, \quad 2 > |z| > 1 \\
y[n] &= \frac{33}{4}u[n-2] - \frac{33}{4}u[n-14] + 16(2)^{n-2}u[-n+1] + 16\left(1 - \frac{1}{5}\left(\frac{1}{3}\right)^{10}\right) 2^{(n-14)} u[-n+13] \\
&- \frac{1}{8}\left(\frac{1}{2}\right)^{n-2} u[n-2] + \frac{1}{8}\left(\left(\frac{1}{3}\right)^{10} + 1\right)\left(\frac{1}{3}\right)^{n-14} u[n-14] \\
&+ \frac{27}{10}\left(\frac{1}{3}\right)^{11}\left(\frac{1}{3}\right)^{n-14} u[n-14]
\end{aligned}
$$

Case I : For $n \leq 1$

$$
\begin{aligned}
y[n] &= 2^{(n+2)} - \left[1 - \frac{1}{5}\left(\frac{1}{3}\right)^{10}\right] 2^{(n-10)} \\
&= \left[4 + \frac{1}{5}\left(\frac{1}{3}\right)^{10} - 2^{-10}\right] 2^n
\end{aligned}
$$

Case II : For $2 \le n \le 13$

$$\begin{aligned} y[n] &= -\left[1 - \frac{1}{5}\left(\frac{1}{3}\right)^{10}\right] 2^{n-10} + \frac{33}{4} - \left(\frac{1}{2}\right)^{n+1} \\ &= \left(\frac{1}{5}\left(\frac{1}{6}\right)^{10} - 2^{-10}\right) 2^{n} - \frac{1}{2}2^{-n} + \frac{33}{4} \end{aligned}$$

Case III : For $n > 13$

$$\begin{aligned} y[n] &= \frac{33}{4} - \left(\frac{1}{2}\right)^{n+1} + \left[\left(\frac{1}{3}\right)^{10} + 1\right]\left(\frac{1}{2}\right)^{n-11} + \frac{1}{10}\left(\frac{1}{3}\right)^{n-6} - \frac{33}{4} \\ &= \left[\left(\frac{1}{2}\right)^{-11}\left(\frac{1}{3}\right)^{10} + \left(\frac{1}{2}\right)^{-11} - \frac{1}{2}\right] 2^{-n} + \frac{729}{10}\left(\frac{1}{3}\right)^{n} \end{aligned}$$

Problem 7
The impulse response is given by taking the inverse z-transform of $H(z)$

$$\begin{aligned} H(z) &= \frac{z^2 - \frac{1}{2}}{z^2 - \frac{3}{4}z + \frac{1}{8}} \\ &= 1 + \frac{\frac{3}{4}z - \frac{5}{8}}{(z - \frac{1}{2})(z - \frac{1}{4})} \\ &= 1 + \frac{7}{4}z^{-1}\frac{z}{z - \frac{1}{4}} - z^{-1}\frac{z}{z - \frac{1}{2}}, \quad |z| > \frac{1}{2} \\ &= 1 + 7\left(\frac{1}{4}\right)^{n} u[n-1] - \left(\frac{1}{2}\right)^{n-1} u[n-1] \end{aligned}$$

The Difference equation can be evaluated as follows,

$$\begin{aligned} \frac{Y(z)}{X(z)} &= \frac{z^2 - \frac{1}{2}}{z^2 - \frac{3}{4}z + \frac{1}{8}} \\ Y(z)\left(z^2 - \frac{3}{4}z + \frac{1}{8}\right) &= X(z)\left(z^2 - \frac{1}{2}\right) \\ y[n+2] - \frac{3}{4}y[n+1] + \frac{1}{8}y[n] &= x[n+2] - \frac{1}{2}x[n] \end{aligned}$$

Problem 8
The pole-zero plot is shown in Fig. 1.

$$\begin{aligned} X(z) &= \frac{z^2 + z + 2}{z^2 + z - 2} \\ &= 1 + \frac{4}{z^2 + z - 2} \\ &= 1 + \frac{4}{(z-1)(z+2)} \\ &= 1 + \frac{\frac{4}{3}}{z - 1} - \frac{\frac{4}{3}}{z + 1} \\ &= 1 + \frac{4}{3}z^{-1}\frac{z}{z - 1} - \frac{4}{3}z^{-1}\frac{z}{z + 2} \end{aligned}$$

Case I : ROC : $|z| > 2$

$$x[n] = \delta[n] + \frac{4}{3}u[n-1] - \frac{4}{3}(-2)^{n-1}u[n-1]$$

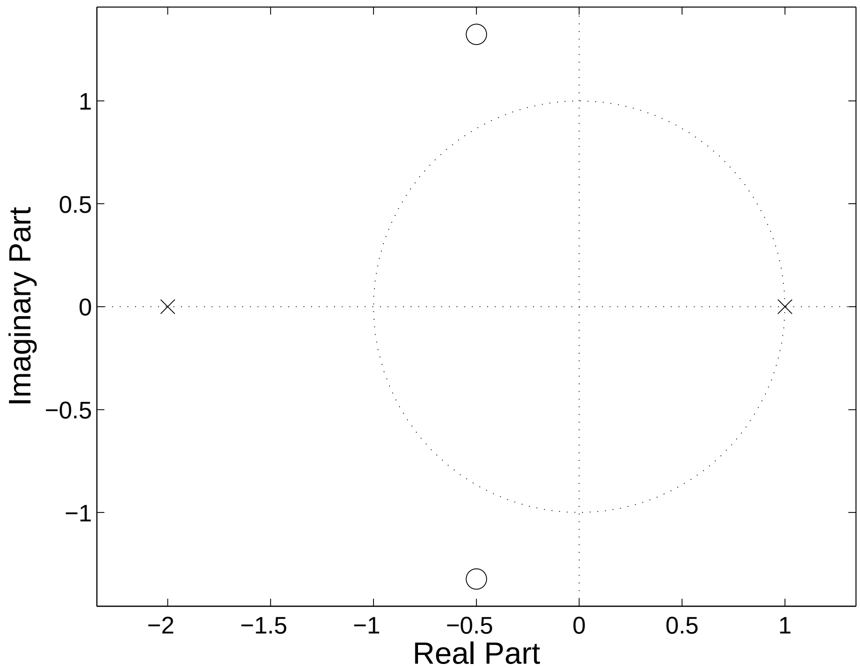

Figure 1: Pole-zero plot for Problem 8

Case II : ROC : $2 > |z| > 1$

$$x[n] = \delta[n] + \frac{4}{3}u[n-1] + \frac{4}{3}(-2)^{n-1}u[-n]$$

Case III : ROC : $|z| < 1$

$$x[n] = \delta[n] - \frac{4}{3}u[-n] + \frac{4}{3}(-2)^{n-1}u[-n]$$

Problem 9

a)

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{\infty} \left(\frac{1}{5}\right)^k u[k] \left(\frac{1}{2}\right)^{n-k} u[n-k-2] \\ &= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n-2} \left(\frac{2}{5}\right)^k \\ &= \left(\frac{1}{2}\right)^n \left[\frac{1-(2/5)^{n-1}}{1-2/5}\right] u[n-2] \\ &= \left[\frac{5}{3}\left(\frac{1}{2}\right)^n - \frac{25}{6}\left(\frac{1}{5}\right)^n\right] u[n-2] \end{aligned}$$

b)

$$
\begin{aligned}
Y(z) &= \frac{z}{z-1/5} \times \frac{1}{4}\frac{z}{z-1/2}z^{-2} \\
\frac{Y(z)}{z} &= \frac{1}{4(z-1/5)(z-1/2)z} \\
&= \frac{A}{z-1/5} + \frac{B}{z-1/2} + \frac{C}{z}
\end{aligned}
$$

We have $A = \frac{-25}{6}$, $B = \frac{5}{3}$, $C = \frac{5}{2}$

$$
\begin{aligned}
y[n] &= \left[\frac{5}{3}\left(\frac{1}{2}\right)^n - \frac{25}{6}\left(\frac{1}{5}\right)^n + \frac{5}{2}\delta[n]\right]u[n] \\
y[n] &= \left[\frac{5}{3}\left(\frac{1}{2}\right)^n - \frac{25}{6}\left(\frac{1}{5}\right)^n\right]u[n-2]
\end{aligned}
$$

Problem 10

taking z-transforms of both sides,

$$
\begin{aligned}
Y(z)[1-0.5z^{-1}+0.04z^{-2}] &= X(z)[1+z^{-1}] \\
H(z) = \frac{Y(z)}{X(z)} &= \frac{1+z^{-1}}{1-0.5z^{-1}+0.04z^{-2}} \\
&= \frac{(z+1)z}{z^2-0.5z+0.04}
\end{aligned}
$$

Consider the input $X(z) = \frac{z}{z-1/4}$, the output is,

$$
\begin{aligned}
Y(z) &= \frac{z}{z-1/4}.\frac{(z+1)z}{(z-0.4)(z-0.1)} \\
\Rightarrow \frac{Y(z)}{z} &= \frac{z(z+1)}{(z-1/4)(z-0.4)(z-0.1)} \\
&= \frac{A}{z-1/4} + \frac{B}{z-0.4} + \frac{C}{z-0.1}
\end{aligned}
$$

Therefore we have $A = 13.9$, $B = 12.45$ and $C = 2.45$. The output $Y(z)$ can now be written as,

$$
\begin{aligned}
Y(z) &= \frac{-13.9z}{z-1/4} + \frac{12.45z}{z-0.4} + \frac{2.45z}{z-0.1} \\
\Rightarrow y[n] &= \left[-13.9(\frac{1}{4})^n + 12.45(0.4)^n + 2.45(0.1)^n\right]u[n]
\end{aligned}
$$

Problem 11

Taking z-transforms of both sides we get the transfer function as,

$$
H(z) = \frac{Y(z)}{X(z)} = \frac{4+8z^{-1}}{1-0.5z^{-1}+0.06z^{-2}}
$$

The impulse response is obtained by taking a inverse z-transform.

$$
\begin{aligned}
\frac{H(z)}{z} &= \frac{4(z+2)}{z^2-0.5z+0.06} \\
&= \frac{A}{z-0.2} + \frac{B}{z-0.3} \\
A = -22, &\quad B = 23 \\
H(z) &= \frac{-88z}{z-0.2} + \frac{92z}{z-0.3} \\
\Rightarrow h[n] &= [-88(0.2^n + 92(0.3)^n)]u[n]
\end{aligned}
$$

Problem 12

$$\begin{aligned} y[n]-(1+r)y[n-1] &= -(1+r)x[n] \\ Y(z)-(1+r)z^{-1}(Y(z)+zy[-1]) &= -(1+r)x[n] \\ Y(z) &= \frac{y[0]-(1+r)X(z)}{1-(1+r)z^{-1}} \end{aligned}$$

$y[-1]=\frac{y[0]}{1+r}=\frac{P}{1+r}$ and $x[n]=Xu[n] \Rightarrow X(z)=\frac{X}{1-z^{-1}}$

$$\begin{aligned} Y(z) &= \frac{P-\frac{(1+r)X}{1+z^{-1}}}{1-(1+r)z^{-1}} \\ &= \frac{\frac{(1+r)^2X}{r}+P}{1-(1+r)z^{-1}}-\frac{\frac{(1+r)X}{r}}{1-z^{-1}} \\ y[n] &= \left(\frac{(1+r)^2X}{r}+P\right)(1+r)^n u[n]-\left(\frac{(1+r)X}{r}\right)u[n] \end{aligned}$$

Use $P=300,000, r=0.08/12$ and the final condition, i.e. the mortgage is paid off in 30 years and you do not owe anything at the end of 30 years, to find X.

$$\begin{aligned} y[30*12+1] &= 0 \\ \left(\frac{(1+r)^2X}{r}+P\right)(1+r)^{360}-\left(\frac{(1+r)X}{r}\right) &= 0 \\ X &= \frac{Pr(1+r)^{360}}{(1+r)^{362}-(1+r)} \\ &= 2170.7 \end{aligned}$$

A monthly payment of $2170.7 is required.