University of Illinois at Urbana-Champaign

ECE 310: Digital Signal Processing

PROBLEM SET 4: SOLUTIONS

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Problem 1

1. $H(z) = \frac{z-7}{z^2+1/9}$, causal
   ROC ($|z| > \frac{1}{3}$) contains the unit circle $\Rightarrow$ BIBO stable

2. $H(z) = \frac{z-7}{z^2+1/9}$, anticausal
   ROC ($|z| < \frac{1}{3}$) doesn't contain the unit circle $\Rightarrow$ NOT BIBO stable. Input example: for $x[n] = \delta[n]$, output $y[n] = -\left(\frac{1}{3}\right)^{n-1} \cos([n-1]\pi/2)u[-n] + 7\left(\frac{1}{3}\right)^{n-2} \cos([n-1]\pi/2)u[-n+1]$ is unbounded.

3. $H(z) = \frac{z}{(z-0.7)(z^2+z+1)}$, $\quad h[n]$ is two-sided
   ROC ($0.7 < |z| < 1$) doesn't contain the unit circle $\Rightarrow$ NOT BIBO stable. Input example: for $x[n] = \sin(2n\pi/3)u[n]$, $X(z) = \frac{\sqrt{\frac{2}{3}}z^2}{z^2+z+1}$ and $Y(z) = \frac{\sqrt{\frac{2}{3}}z^3}{(z-0.7)(z^2+z+1)^2}$. Therefore, $y[n]$ contains a term $ne^{j\frac{2}{3}n\pi}$, which is unbounded.

4. $H(z) = \frac{z+1}{z-1}$, causal
   ROC ($|z| > 1$) doesn't contain the unit circle $\Rightarrow$ NOT BIBO stable. Input example: for $x[n] = u[n]$, $Y(z) = \frac{z^2+z}{(z-1)^2}$. Therefore, $y[n]$ contains a term of the form $nu[n]$, which is unbounded.

5. $H(z) = \frac{z-1}{z^2+j}$, causal
   ROC ($|z| > 1$) doesn't contain the unit circle $\Rightarrow$ NOT BIBO stable. Input example: for $x[n] = \cos(n\pi/4)u[n]$, $Y(z)$ has double poles at $e^{-j\pi/4}$. Therefore, $y[n]$ contains a term of the form $ne^{-j(\pi/4)n}u[n]$, which is unbounded.

Problem 2

(a) Let the output of the two adders be $u[n]$ and $v[n]$,

$$\begin{aligned}
U(z) &= 4z^{-1}(zX(z) + 3V(z)) \\
\Rightarrow U(z) &= \frac{(8z+180)X(z)}{z^2-120} \\
V(z) &= 5z^{-1}(3X(z) + 2U(z)) \\
\Rightarrow V(z) &= \frac{(15z+80)}{z^2-120}X(z) \\
\Rightarrow Y(z) &= U(z) + V(z) \\
&= \frac{23z+260}{z^2-120}X(z) \\
\Rightarrow Y(z)(z^2-120) &= X(z)(23z+260) \\
\Rightarrow y[n+2] - 120y[n] &= 23x[n+1] + 260x[n]
\end{aligned}$$

(b)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{23z+260}{z^2-120} = \frac{23z^{-1}+260z^{-2}}{1-120z^{-2}}$$

The direct form structure is shown in Fig.

Problem 3

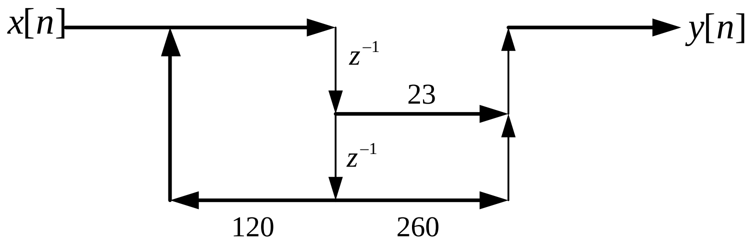

Figure 1: Direct Form II structure for Problem 2(b)

a) Let $m = n + 2$, then

$$0 = 2y[m] - 2y[m-2] + 4x[m] + 2x[m-1]$$
$$\therefore\ y[n] = y[n-2] - 2x[n] - x[n-1]$$

b) Two possible block diagrams are given below:

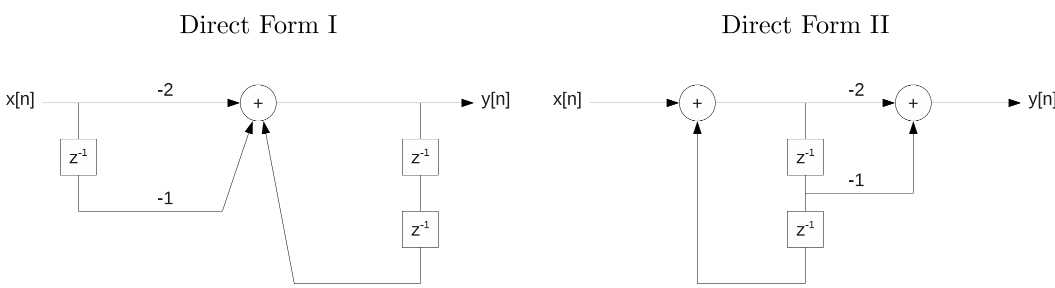

Figure 2: Filter structure for Problem 3(b)

c) For the LSI system, the output $y[n]$ is the impulse response $h[n]$ when $x[n] = \delta[n]$:

$$h[n] - h[n-2] = -2\delta[n] - \delta[n-1]$$

Using the zero initial conditions ($h[-2] = h[-1] = 0$),

$$\begin{aligned}
n = 0:&\quad h[0] - h[-2] = -2 \longrightarrow h[0] = -2\\
n = 1:&\quad h[1] - h[-1] = -1 \longrightarrow h[1] = -1\\
n \geq 2:&\quad h[n] - h[n-2] = 0\\
&\quad z^2 - 1 = 0\\
&\quad h[n] = A(-1)^n + B(1)^n\\
&\quad h[0] = A + B = -2\\
&\quad h[1] = -A + B = -1\\
&\quad A = -\tfrac{1}{2},\ B = -\tfrac{3}{2}\\
&\quad \therefore\ h[n] = \left(-\tfrac{1}{2}(-1)^n - \tfrac{3}{2}\right) u[n]
\end{aligned}$$

Therefore,

$$y[-1] = 0,\ y[0] = -2,\ y[1] = -1,\ y[2] = -2,\ y[3] = -1$$

d) From $y[n] = y[n-2] - 2x[n] - x[n-1]$ in (a),

$$Y(z)(1 - z^{-2}) = X(z)(-2 - z^{-1}) \longrightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{-2 - z^{-1}}{1 - z^{-2}}$$

e)

$$
\begin{aligned}
H(z) &= \frac{-2z^2 - z}{z^2 - 1} \\
\frac{H(z)}{z} &= \frac{-2z - 1}{(z+1)(z-1)} = \frac{A}{z+1} + \frac{B}{z-1} \\
A &= \left.\frac{-2z-1}{z-1}\right|_{z=-1} = -\frac{1}{2} \\
B &= \left.\frac{-2z-1}{z+1}\right|_{z=1} = -\frac{3}{2} \\
H(z) &= -\frac{1}{2} \cdot \frac{z}{z+1} - \frac{3}{2} \cdot \frac{z}{z-1} \text{ where system is causal} \\
\therefore\ h[n] &= \left(-\frac{1}{2}(-1)^n - \frac{3}{2}\right) u[n]
\end{aligned}
$$

Therefore,

$$h[-1] = 0,\ h[0] = -2,\ h[1] = -1,\ h[2] = -2,\ h[3] = -1$$

which is identical to the samples of the unit pulse response computed above via direct iteration of the difference equation.

Problem 4

1.

$$
\begin{aligned}
y[n] &= x[n] + x[n-10] \\
Y(z) &= (1 + z^{-10})X(z) \\
\therefore H(z) &= 1 + z^{-10} \\
H_d(\omega) &= 1 + e^{-j10\omega} = 2e^{-j5\omega} \cos(5\omega) \\
|H_d(\omega)| &= 2|\cos(5\omega)| \\
\angle H_d(\omega) &= \begin{cases} -5\omega & \text{for } \cos(5\omega) \geq 0 \\ -5\omega + \pi & \text{for } \cos(5\omega) < 0 \end{cases}
\end{aligned}
$$

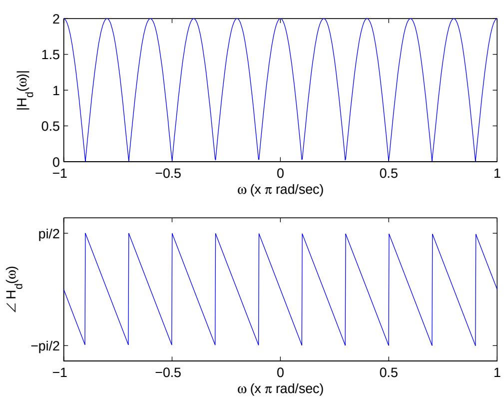

Figure 3: Magnitude and Phase for Problem 4

2. Note $h[n]$ is real since $H_d(\omega) = H_d^*(-\omega)$.

(a)

$$x[n] = \cos\left(\frac{\pi}{10}n\right) + 3\sin\left(\frac{\pi}{3}n + \frac{\pi}{10}\right)$$
$$\therefore y[n] = |H_d(\frac{\pi}{10})|\cos\left(\frac{\pi}{10}n + \angle H_d(\frac{\pi}{10})\right) + 3|H_d(\frac{\pi}{3})|\sin\left(\frac{\pi}{3}n + \frac{\pi}{10} + \angle H_d(\frac{\pi}{3})\right)$$
$$y[n] = 3\sin\left(\frac{\pi}{3}n + \frac{\pi}{10} + \frac{\pi}{3}\right)$$
$$y[n] = 3\sin\left(\frac{\pi}{3}n + \frac{13\pi}{30}\right)$$

(b)

$$x[n] = 10 + 5\cos\left(\frac{2\pi}{5}n + \frac{\pi}{2}\right)$$
$$\therefore y[n] = 10H_d(0) + 5|H_d(\frac{2\pi}{5})|\cos\left(\frac{2\pi}{5}n + \frac{\pi}{2} + \angle H_d(\frac{2\pi}{5})\right)$$
$$y[n] = 20 + 10\cos\left(\frac{2\pi}{5}n + \frac{\pi}{2}\right)$$

Problem 5

$$\begin{aligned}
h_1[n] &= \frac{4}{7}(2)^{n-1}u[-n] + \frac{11}{7}\left(\frac{1}{4}\right)^{n-1}u[n-1] \\
&= \frac{2}{7}(2)^{n}u[-n] + \frac{44}{7}\left(\frac{1}{4}\right)^{n}u[n-1] \\
&= \frac{2}{7}(2)^{n}u[-n-1] + \frac{2}{7}\delta[n] + \frac{44}{7}\left(\frac{1}{4}\right)^{n}u[n] - \frac{44}{7}\delta[n] \\
&= -\frac{2}{7}(-(2)^{n}u[-n-1]) + \frac{44}{7}\left(\frac{1}{4}\right)^{n}u[n] - 6\delta[n] \\
H_1(z) &= -6 - \frac{2}{7}\left(\frac{z}{z-2}\right) + \frac{44}{7}\left(\frac{z}{z-\frac{1}{4}}\right) \ , \ \frac{1}{4} < |z| < 2 \\
&= \frac{\frac{44}{7}z(z-2) - \frac{2}{7}z(z-\frac{1}{4}) - 6(z-2)(z-\frac{1}{4})}{(z-2)(z-\frac{1}{4})} \ , \ \frac{1}{4} < |z| < 2 \\
&= \frac{z-3}{(z-2)(z-\frac{1}{4})} \ , \ \frac{1}{4} < |z| < 2
\end{aligned}$$

$$\begin{aligned}
h_2[n] &= \delta[n] - 3^{n-1}u[-n] = \delta[n] - \frac{1}{3}3^{n}u[-n] \\
&= \delta[n] - \frac{1}{3}3^{n}u[-n-1] - \frac{1}{3}\delta[n] \\
&= \frac{2}{3}\delta[n] + \frac{1}{3}(-(3)^{n}u[-n-1]) \\
H_2(z) &= \frac{2}{3} + \frac{1}{3}\left(\frac{z}{z-3}\right) \ , \ |z| < 3 \\
&= \frac{z-2}{z-3} \ , \ |z| < 3 \\
H_1(z)H_2(z) &= \frac{(z-3)(z-2)}{(z-2)(z-\frac{1}{4})(z-3)} \ , \ \frac{1}{4} < |z| < 2 \\
&= z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right) \ , \ |z| > \frac{1}{4}
\end{aligned}$$

$H_1(z)$ is not causal, BIBO stable
$H_2(z)$ is not causal, BIBO stable
$H_1(z)H_2(z)$ is causal, BIBO stable.

Problem 6

The inputs

$$x_1[n] = 3, \; x_2[n] = \cos\left(\frac{\pi}{4}n + 10^\circ\right) = \sin\left(\frac{\pi}{4}n + 100^\circ\right), \; x_3[n] = \sin\left(\frac{\pi}{3}n + 25^\circ\right)$$

correspond to outputs

$$y_1[n] = 9, \; y_2[n] = 2\sin\left(\frac{\pi}{4}n + 10^\circ\right), \; y_3[n] = 0$$

Therefore,

$$H_d(0) = 3 \qquad H_d(\frac{\pi}{4}) = 2e^{-j90^\circ} \qquad H_d(\frac{\pi}{3}) = 0$$

$$\begin{aligned}
\tilde{x}[n] &= 5 + 2\sin\left(\frac{\pi}{4}n + 15^\circ\right) + 10\cos\left(-\frac{\pi}{3}n + 25^\circ\right) \\
&= 5 + 2\sin\left(\frac{\pi}{4}n + 15^\circ\right) + 10\cos\left(\frac{\pi}{3}n - 25^\circ\right) \\
\therefore \tilde{y}[n] &= 5H_d(0) + 2|H_d(\frac{\pi}{4})|\sin\left(\frac{\pi}{4}n + 15^\circ + \angle H_d(\frac{\pi}{4})\right) + 10|H_d(\frac{\pi}{3})|\cos\left(\frac{\pi}{3}n - 25^\circ + \angle H_d(\frac{\pi}{3})\right) \\
&= 15 + 4\sin\left(\frac{\pi}{4}n - 75^\circ\right)
\end{aligned}$$

Problem 7

Note $h[n]$ is NOT real since $H_d(\omega) \neq H_d^*(-\omega)$.

1.

$$\begin{aligned}
x[n] &= 5 + 10e^{j(\frac{\pi}{4}n + 45^\circ)} + j^n \\
&= 5 + 10e^{j(\frac{\pi}{4}n + \frac{\pi}{4})} + e^{j\frac{\pi}{2}n} \\
\therefore y[n] &= 5H_d(0) + 10e^{j(\frac{\pi}{4}n + \frac{\pi}{4})}H_d(\frac{\pi}{4}) + e^{j\frac{\pi}{2}n}H_d(\frac{\pi}{2}) \\
&= 0 + 10e^{j(\frac{\pi}{4}n + \frac{\pi}{4})} \cdot \frac{\pi}{4}e^{\frac{j}{\sqrt{2}}} + e^{j\frac{\pi}{2}n} \cdot \frac{\pi}{2}e^{j} \\
&= \frac{5\pi}{2}e^{j(\frac{\pi}{4}n + \frac{\pi}{4} + \frac{\sqrt{2}}{2})} + \frac{\pi}{2}e^{j(\frac{\pi}{2}n + 1)}
\end{aligned}$$

2.

$$\begin{aligned}
x[n] &= 5 + 10\cos(\frac{\pi}{4}n + 45^\circ) + j^n \\
&= 5 + 5e^{j(\frac{\pi}{4}n + \frac{\pi}{4})} + 5e^{-j(\frac{\pi}{4}n + \frac{\pi}{4})} + e^{j\frac{\pi}{2}n} \\
\therefore y[n] &= 5H_d(0) + 5e^{j(\frac{\pi}{4}n + \frac{\pi}{4})}H_d(\frac{\pi}{4}) + 5e^{-j(\frac{\pi}{4}n + \frac{\pi}{4})}H_d(-\frac{\pi}{4}) + e^{j\frac{\pi}{2}n}H_d(\frac{\pi}{2}) \\
&= 0 + 5e^{j(\frac{\pi}{4}n + \frac{\pi}{4})} \cdot \frac{\pi}{4}e^{\frac{j}{\sqrt{2}}} - 5e^{-j(\frac{\pi}{4}n + \frac{\pi}{4})} \cdot \frac{\pi}{4}e^{-\frac{j}{\sqrt{2}}} + e^{j\frac{\pi}{2}n} \cdot \frac{\pi}{2}e^{j} \\
&= \frac{5j\pi}{2}\sin\left(\frac{\pi}{4}n + \frac{\pi}{4} + \frac{\sqrt{2}}{2}\right) + \frac{\pi}{2}e^{j(\frac{\pi}{2}n + 1)}
\end{aligned}$$

Problem 8

(a)

$$
\begin{aligned}
Y(z) &= \frac{1}{1-3z^{-1}} - \frac{1}{1-2z^{-1}} \ , \ |z| > 3 \\
&= \frac{z^{-1}}{(1-2z^{-1})(1-3z^{-1})} \ , \ |z| > 3 \\
X(z) &= \frac{1}{1-2z^{-1}} - \frac{6z^{-1}}{1-2z^{-1}} \ , \ |z| > 2 \\
&= \frac{1-6z^{-1}}{1-2z^{-1}} \ , \ |z| > 2 \\
H(z) &= \frac{z^{-1}}{(1-3z^{-1})(1-6z^{-1})} \ , \ \text{subject to } |z| > 3 \\
&= \frac{1}{3}\left(\frac{1}{1-6z^{-1}} - \frac{1}{1-3z^{-1}}\right) \ , \ \text{subject to } |z| > 3
\end{aligned}
$$

One-sided, causal solution (ROC: $|z| > 6$):

$$h[n] = \left(-\frac{1}{3}(3)^n + \frac{1}{3}(6)^n\right) u[n]$$

Two-sided solution (ROC: $3 < |z| < 6$):

$$h[n] = -\frac{1}{3}(3)^n u[n] - \frac{1}{3}(6)^n u[-n-1]$$

(b) The solution is not unique. There are two possible answers as shown above. The system is unstable for both solutions. Therefore, even if the system is known to be unstable, there are two possible solutions. If the system is given to be causal, then the ROC is $|z| > 6$ and there is a unique, one-sided solution for $h[n]$.

Problem 9

(a) The poles are at $re^{\pm j\omega_0}$ and zeros at $e^{\pm j\omega_0}$. The sketch is shown in Fig. 4

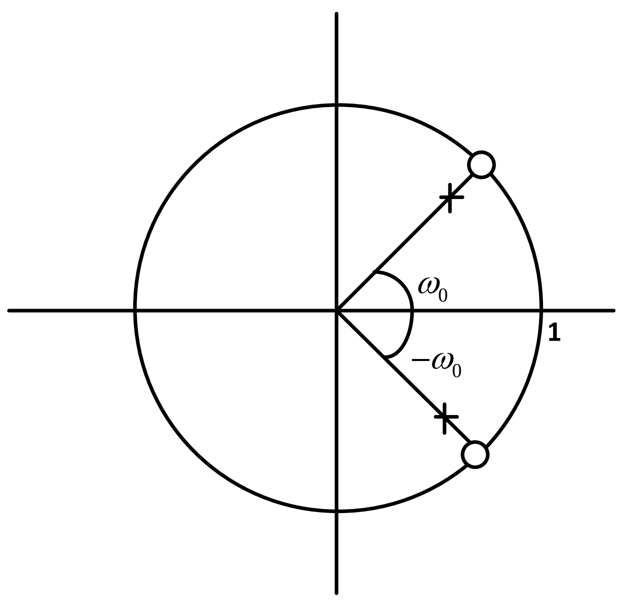

Figure 4: Pole-zero plot for Problem 9(a)

(b) For $\omega = \omega_0$, $H_d(\omega_0) = 0$. For values of $\omega \neq 0$, the poles and zeros cancel resulting in $H_d(\omega) = 1$.

(c)

$$|H(\omega)|^2 = G^2 \frac{|1 - e^{j\omega_0}e^{-j\omega}|^2 |1 - e^{-j\omega_0}e^{-j\omega}|^2}{|1 - re^{-j\omega_0}e^{-j\omega}|^2 |1 - re^{-j\omega_0}e^{-j\omega}|^2}$$
$$= G^2 \left[\frac{2(1 - cos(\omega - \omega_0))}{1 + r^2 - 2rcos(\omega - \omega_0)}\right] \left[\frac{2(1 - cos(\omega + \omega_0))}{1 + r^2 - 2rcos(\omega + \omega_0)}\right]$$

where $\omega_0 = \frac{\pi}{3}$. Then

$$\frac{d|H(\omega)|}{d\omega} = 0 \Rightarrow \omega = \pi$$
$$\Rightarrow |H(\pi)|^2 = 4G^2 \left(\frac{\frac{3}{2}}{1 + r + r^2}\right) = 1$$
$$G = \frac{1}{3}(1 + r + r^2)$$

(d) The direct form II structure is shown in Fig. 5.

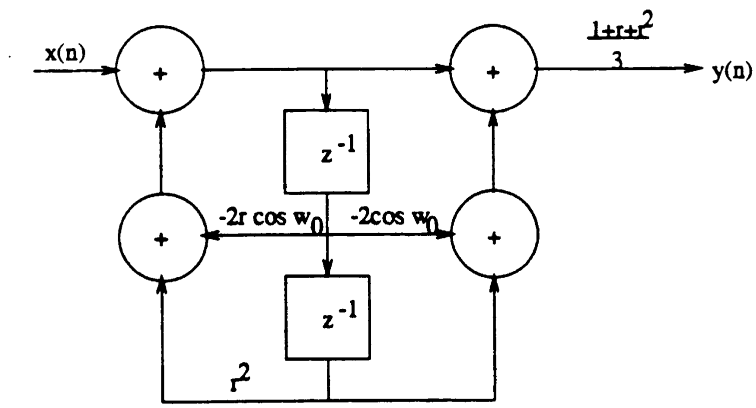

Figure 5: Pole-zero plot for Problem 9(d)

Problem 10
All three systems are causal.

(a)

$$H_1(z) = \frac{3}{2}Y(z) - \frac{7}{2}z^{-1}Y(z) + z^{-2}Y(z) = 2X(z) + z^{-1}X(z)$$
$$= \frac{\frac{4}{3}z(z + \frac{1}{2})}{z^2 - \frac{7}{3}z + \frac{2}{3}}$$
$$= \frac{\frac{4}{3}z(z + \frac{1}{2})}{(z - 2)(z - \frac{1}{3})}$$

Hence ROC : $|z| > 2$. The system is not BIBO stable.
Similarly,

$$H_2(z) = \frac{(z - \frac{1}{4})(z - \frac{1}{3})}{4z(z + \frac{1}{2})}$$

ROC: $|z| > \frac{1}{2}$, BIBO stable.

$$H_3(z) = \frac{(-2z(z-2)}{(z+\frac{1}{5})(z-\frac{1}{2})}$$

ROC: $|z| > \frac{1}{2}$. BIBO stable.

(b) Since $H_2(z)$ and $H_3(z)$ are BIBO stable we can connect them in series in any order to create a BIBO stable system.

$$H_2(z)H_3(z) = \frac{-(z-2)(z-\frac{1}{4})(z-\frac{1}{3})}{2(z+\frac{1}{2})(z+\frac{1}{5})(z-\frac{1}{2})}$$

Note: We can also have cascade of $H_1(z)$ and $H_3(z)$. In this case $H_3(z)$ must be the first system in the cascade.

$$H_1(z)H_3(z) = \frac{\frac{-8}{3}z^2(z+\frac{1}{2})}{(z-\frac{1}{3})(z+\frac{1}{3})(z-\frac{1}{2})}$$

ROC: $|z| > \frac{1}{2}$

(c)

$$\begin{aligned}
Y(z)(z+\frac{1}{5})(z-\frac{1}{2})(z-\frac{1}{3}) &= -X(z)\frac{8}{3}z^2(z+\frac{1}{2}) \\
Y(z)\left(z^3 - \frac{19}{30}z^2 + \frac{1}{30}\right) &= -X(z)\left(\frac{8}{3}z^3 + \frac{4}{3}z^2\right) \\
\Rightarrow y[n] - \frac{19}{30}y[n-1] + \frac{1}{30}y[n-3] &= \frac{-8}{3}X[n] - \frac{4}{3}X[n-1]
\end{aligned}$$

Problem 11

(a) By looking at the plots, the only system that is BIBO stable is system C, whose poles all lie within the unit circle. The other systems have poles have outside the unit circle.

(b)

$$\begin{aligned}
H_A(z) &= \frac{(z-2)(z+2)}{(z-0.5)(z-2e^{j\frac{\pi}{3}})(z-2e^{-j\frac{\pi}{3}})} \\
H_B(z) &= \frac{(z-2)(z+2)(z-0.5)}{(z-2e^{j\frac{2\pi}{3}})(z-2e^{-j\frac{2\pi}{3}})(z-2e^{j\frac{\pi}{3}})(z-2e^{-j\frac{\pi}{3}})} \\
H_C(z) &= \frac{(z-2e^{j\frac{2\pi}{3}})(z-2e^{-j\frac{2\pi}{3}})(z-2e^{j\frac{\pi}{3}})(z-2e^{-j\frac{\pi}{3}})}{(z-0.5e^{j\frac{2\pi}{3}})(z-0.5e^{-j\frac{2\pi}{3}})(z-0.5e^{j\frac{\pi}{3}})(z-0.5e^{-j\frac{\pi}{3}})} \\
H_D(z) &= \frac{(z-0.5)(z+0.5)}{(z-2)(z+2)}
\end{aligned}$$

(c)

$$\begin{aligned}
H_{BC}(z) &= H_B(z)H_C(z) \\
&= \frac{(z-2)(z+2)(z-0.5)}{(z-0.5e^{j\frac{2\pi}{3}})(z-0.5e^{-j\frac{2\pi}{3}})(z-0.5e^{j\frac{\pi}{3}})(z-0.5e^{-j\frac{\pi}{3}})}
\end{aligned}$$

Since the poles are inside the unit circle the system is BIBO stable.

Now compute the inverse z-transform,

$$\begin{aligned}
H_{BC}(z) &= \frac{\frac{9}{2} - j4\sqrt{3}}{z - 0.5e^{j\frac{2\pi}{3}}} + \frac{\frac{9}{2} + j4\sqrt{3}}{z - 0.5e^{-j\frac{2\pi}{3}}} \\
&+ \frac{-4 + j\frac{3\sqrt{3}}{2}}{z - 0.5e^{j\frac{\pi}{3}}} + \frac{-4 - j\frac{3\sqrt{3}}{2}}{z - 0.5e^{-j\frac{\pi}{3}}} \\
\Rightarrow h_{BC}[n] &= \left(\frac{9}{2} - j4\sqrt{3}\right)\left(0.5e^{j\frac{2\pi}{3}}\right)^{n-1} u[n-1] + \left(\frac{9}{2} + j4\sqrt{3}\right)\left(0.5e^{-j\frac{2\pi}{3}}\right)^{n-1} u[n-1] \\
&+ \left(-4 + j\frac{3\sqrt{3}}{2}\right)\left(0.5e^{j\frac{\pi}{3}}\right)^{n-1} u[n-1] + \left(-4 - j\frac{3\sqrt{3}}{2}\right)\left(0.5e^{-j\frac{\pi}{3}}\right)^{n-1} u[n-1]
\end{aligned}$$

(d)

$$H_{AD}(z) = H_A(z)H_D(z) = \frac{z + 0.5}{(z - 2e^{j\frac{\pi}{3}})(z - 2e^{j\frac{\pi}{3}})}$$

Since the poles are outside the unit circle the system is NOT BIBO stable.
The inverse z-transform is,

$$\begin{aligned}
H_{AD}(z) &= \frac{\frac{1}{2} - j\frac{\sqrt{3}}{4}}{z - 2e^{j\frac{\pi}{3}}} + \frac{\frac{1}{2} + j\frac{\sqrt{3}}{4}}{(z - 2e^{j\frac{\pi}{3}})} \\
\Rightarrow h_{AD}[n] &= \left(\frac{1}{2} - j\frac{\sqrt{3}}{4}\right)\left(2e^{j\frac{\pi}{3}}\right)^{n-1} u[n-1] + \left(\frac{1}{2} + j\frac{\sqrt{3}}{4}\right)\left(2e^{-j\frac{\pi}{3}}\right)^{n-1} u[n-1]
\end{aligned}$$