University of Illinois at Urbana-Champaign

ECE 310: Digital Signal Processing

PROBLEM SET 5: SOLUTIONS

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Problem 1

To derive $x_a(t)$ $(X_a(\Omega))$ from $X_d(\omega)$, we first need to get rid of the repeated frequency component in $X_d(\omega)$. Thus, multiply an ideal LPF on both sides.

$$G_a(\Omega) = \begin{cases} T & |\Omega| < \frac{\pi}{T} \\ 0 & \text{else} \end{cases}$$
$$\Rightarrow g_a(t) = sinc\Big(\frac{\pi}{T}t\Big)$$

Therefore,

$$\Rightarrow X_a\left(\frac{\omega}{T}\right) = X_d(\omega)G_a(\Omega)$$
$$\Rightarrow X_a(\Omega) = X_d(\omega)G_a(\Omega)$$
$$\Rightarrow x_a(t) = x(nT) * g_a(t)$$
$$\Rightarrow \boxed{x_a(t) = \sum_{n=-\infty}^{\infty} x[n] sinc\Big(\frac{\pi}{T}(t-nT)\Big)}$$

Problem 2

(a) The Nyquist sampling rate is given by,

$$T_{Nyquist} = \frac{1}{(12 \cdot 10^3)(2)} = 0.000042 \text{ sec} = 0.042 \text{ ms}$$
$$\therefore\ f_{Nyquist} = \frac{1}{T_{Nyquist}} = 24 \text{ kHz}$$

(b)

$$\omega_{max} = T_{Nyquist} \cdot (2\pi)(6000) = \frac{\pi}{2}$$
$$\omega_{min} = T_{Nyquist} \cdot (2\pi)(300) = \frac{\pi}{40}$$

The sketch of the frequency response of the discrete-time filter, when sampling at the Nyquist rate is shown in Fig. 1

(c) Some aliasing of the input signal is allowed with the condition that the minimum aliasing frequency is greater than the cutoff frequency of the filter.

$$2\pi - 2\pi \cdot 12000 T_{max1} \geq 2\pi \cdot 6000 T_{max1}$$
$$1 \geq 18000 T_{max1}$$
$$T_{max1} = \frac{1}{18000} \text{ sec}$$

Also, the maximum frequency of $H_{d,2}(\omega)$ should be less than or equal to π.

$$\omega_{max} = T_{max2}(2\pi)(6000) \leq \pi$$
$$T_{max2} = \frac{1}{12000} \text{ sec}$$

Therefore, $T = \min(T_{max1}, T_{max2}) = T_{max1} = \frac{1}{18000}$ sec

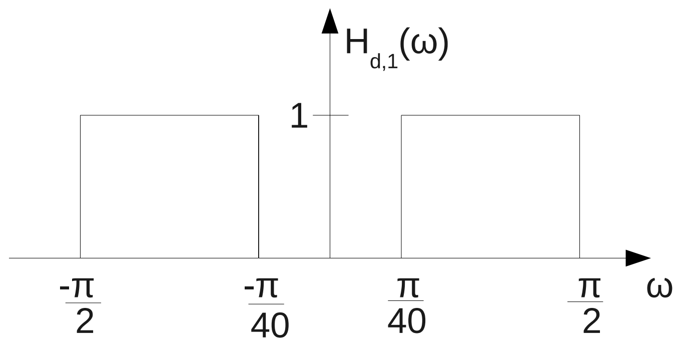

Figure 1: Figure for Problem 2(b)

(d)

$$\omega_{max} = \frac{1}{18000} \cdot (2\pi)(6000) = \frac{2\pi}{3}$$
$$\omega_{min} = \frac{1}{18000} \cdot (2\pi)(300) = \frac{\pi}{30}$$

The sketch of the frequency response of the discrete-time filter, when sampling at the maximum rate is shown in Fig. 2

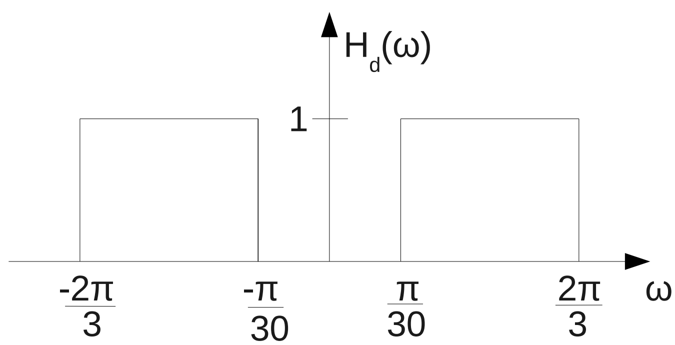

Figure 2: Figure for Problem 2(d)

Problem 3

(a) The Nyquist rate is twice the highest frequency component, $f_s \geq 10$ kHz. Therefore, $T_{max} = \frac{1}{10000}$ sec.

(b)

$$\omega = \Omega T$$
$$\frac{\pi}{8} = \frac{1}{10000}\Omega$$
$$\Omega = 2\pi \cdot 625$$
$$f = 625 \text{ Hz}$$

(c)

$$\omega = \Omega T$$
$$\frac{\pi}{8} = \frac{1}{20000}\Omega$$
$$\Omega = 2\pi \cdot 1250$$
$$f = 1250 \text{ Hz}$$

Problem 4

(a) The sketches for $X_d(\omega)$, $Y_d(\omega)$, and $Y_a(\Omega)$ are given below

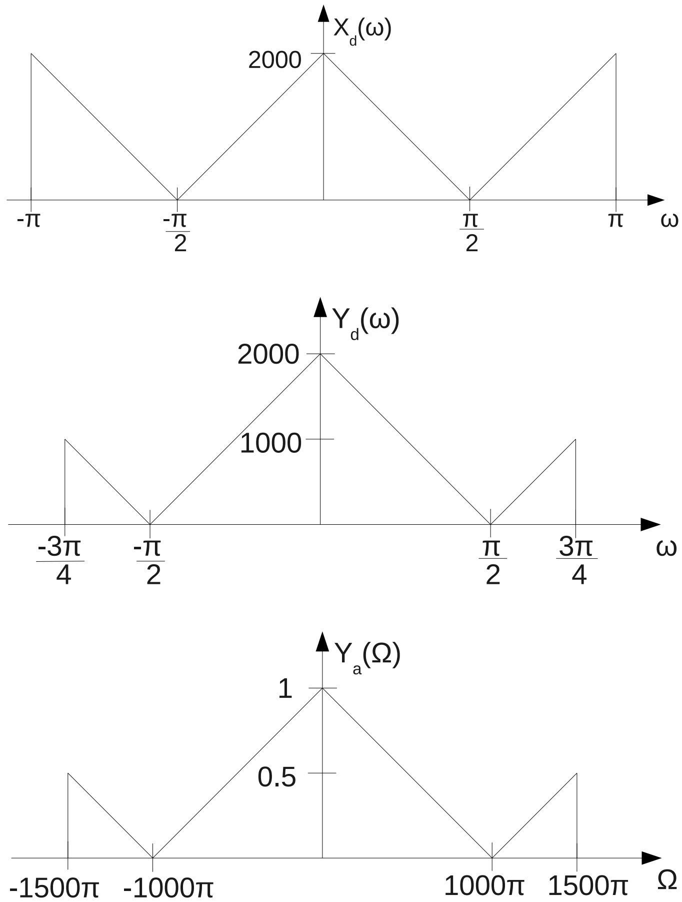

Figure 3: Figure for Problem 4(a)

(b)

$$
\begin{aligned}
P_a(\Omega) &= \int_0^T 1 \cdot e^{-j\Omega t} dt \\
&= T \cdot e^{-j\frac{\Omega T}{2}} sinc\left(\frac{\Omega T}{2}\right) \\
\therefore\ Y_a(\Omega) &= Te^{-j\frac{\Omega T}{2}} sinc\left(\frac{\Omega T}{2}\right) Y_d(\Omega T)
\end{aligned}
$$

The sketch of the magnitude of $Y_a(\Omega)$ is given in Fig. 4: The component of $Y_a(\Omega)$ for $|\Omega| > \frac{\pi}{T} = 2000\pi$ is due to the nonideal D/A. The highest amplitude of this unwanted component of $Y_a(\Omega)$ is at $\Omega = 2500\pi$ rad/sec and

$$
|Y_a(2500\pi)| = Y_d\left(\frac{5\pi}{4}\right) \cdot T \cdot sinc\left(\frac{2500\pi T}{2}\right) = 0.2353
$$

as shown in the figure above.

Problem 5

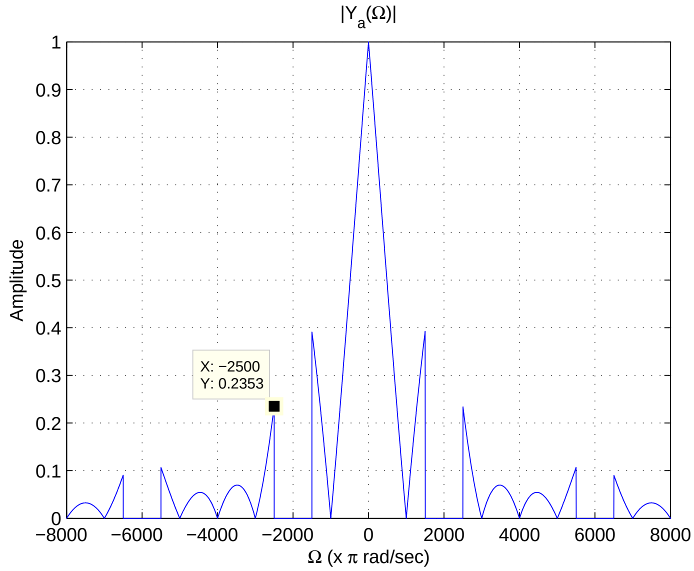

Figure 4: Figure for Problem 4(b)

(a)

$$\begin{aligned} X_0(\Omega) &= P_0(\Omega)X_d(\Omega T) \\ X_1(\Omega) &= P_1(\Omega)X_d(\Omega T) \\ \therefore\ H(\Omega) &= \frac{X_1(\Omega)}{X_0(\Omega)} = \frac{P_1(\Omega)}{P_0(\Omega)} \end{aligned}$$

For zero-order hold:

$$P_0(\Omega) = \int_0^T 1 \cdot e^{-j\Omega t} dt = Te^{-j\frac{\Omega T}{2}} sinc\left(\frac{\Omega T}{2}\right)$$

For first-order hold:

$$\begin{aligned} p_1(t) &= \frac{1}{T} rect\left(\frac{t}{T}\right) * rect\left(\frac{t}{T}\right) = \frac{1}{\sqrt{T}} p_0\left(t + \frac{T}{2}\right) * \frac{1}{\sqrt{T}} p_0\left(t + \frac{T}{2}\right) \\ \therefore\ P_1(\Omega) &= \left(\frac{1}{\sqrt{T}} Te^{-j\frac{\Omega T}{2}} sinc\left(\frac{\Omega T}{2}\right) e^{j\frac{\Omega T}{2}}\right)^2 = T sinc^2\left(\frac{\Omega T}{2}\right) \\ \therefore\ H(\Omega) &= \frac{T sinc^2\left(\frac{\Omega T}{2}\right)}{Te^{-j\frac{\Omega T}{2}} sinc\left(\frac{\Omega T}{2}\right)} = e^{j\frac{\Omega T}{2}} sinc\left(\frac{\Omega T}{2}\right) \end{aligned}$$

(b) For an ideal D/A, $Y_a(\Omega) = G_{ideal}(\Omega) Y_d(\Omega T)$ where:

$G_{ideal}(\Omega) = \begin{cases} T, & |\Omega| \le \frac{\pi}{T} \\ 0, & \text{otherwise} \end{cases}$ For the FOH, to form an ideal D/A, $Y_a(\Omega) = F_1(\Omega) P_1(\Omega) Y_d(\Omega T)$ where $F_1(\Omega)$ is an analog filter that follows the FOH and

$$F_1(\Omega) = \begin{cases} \frac{T}{P_1(\Omega)} = \frac{1}{sinc^2\left(\frac{\Omega T}{2}\right)}, & |\Omega| \le \frac{\pi}{T} \\ 0, & \text{otherwise} \end{cases}$$

(c) Suppose $F_0(\Omega)$ and $F_1(\Omega)$ are the analog filters that follow the ZOH and FOH, respectively. For reference, the frequency responses of the ZOH and FOH and their subsequent filters are given in Fig. 5. The cutoff frequency for both $F_0(\Omega)$ and $F_1(\Omega)$ is $\Omega = \frac{\pi}{T}$. The magnitudes of these LPFs are

$$\begin{aligned} |F_0(\Omega)| &= \left|\frac{1}{sinc\left(\frac{\Omega T}{2}\right)}\right| \\ |F_1(\Omega)| &= \left|\frac{1}{sinc^2\left(\frac{\Omega T}{2}\right)}\right| \end{aligned}$$

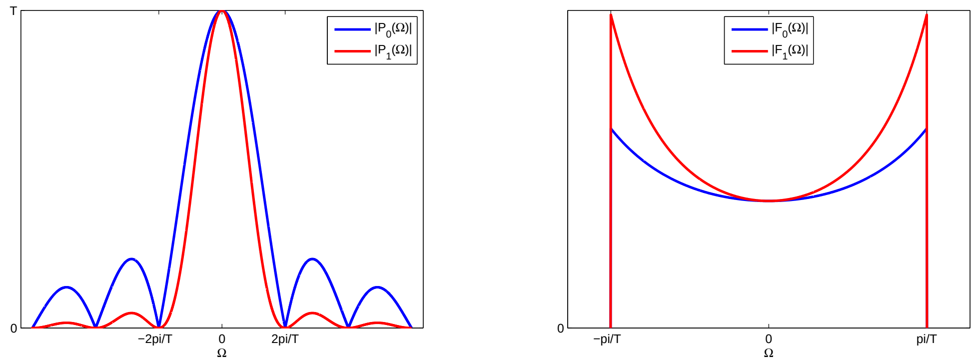

Figure 5: Figure for Problem 5(c)

$|F_1(\Omega)|$ has a frequency response which is steeper that $|F_0(\Omega)|$. Therefore, the FOH (linear interpolation) might interpolate $y[n]$ more precisely than a ZOH (piecewise constant interpolation) at a cost of a more complicated/expensive analog filter that follows the FOH.

Problem 6

(a) It is given that

$$y_a(t) = x_a(t) + \alpha\, x_a(t - \tau_d) + \beta\, x_a(t - 2\tau_d)$$

Applying the Fourier transform on both sides, we have:

$$Y_a(\Omega) = X_a(\Omega) + \alpha X_a(\Omega) e^{-j\Omega\tau_d} + \beta X_a(\Omega) e^{-j2\Omega\tau_d}$$

$$H_a(\Omega) = \frac{Y_a(\Omega)}{X_a(\Omega)} = 1 + \alpha e^{-j\Omega\tau_d} + \beta e^{-j2\Omega\tau_d}$$

(b) Taking appropriate sampling period to mean the Nyquist sampling period:

$$T_0 = \frac{1}{2f_{max}} = \frac{1}{2 \times 20 \times 10^3} = 25\ \mu s$$

(c) With the sampling period $T_0 = 25\ \mu s$ there is no aliasing in the system. With an ideal D/C, the digital filter we need is:

$$\begin{aligned} H_d(\omega) &= H_a(\frac{\omega}{T_0}) = 1 + \alpha e^{-j\omega\tau_d/T_0} + \beta e^{-j2\omega\tau_d/T_0} \qquad (*) \\ &= 1 + \alpha e^{-j40000\omega\tau_d} + \beta e^{-j80000\omega\tau_d} \end{aligned}$$

(d) With $\tau_d = 100T_0$, Equation $(*)$ above simplifies as:

$$H_d(\omega) = 1 + \alpha e^{-100j\omega} + \beta e^{-200j\omega}$$

By substituting $z = e^{j\omega}$, we obtain the following transfer function:

$$H(z) = 1 + \alpha z^{-100} + \beta z^{-200},$$

which can be implemented using the block diagram below.

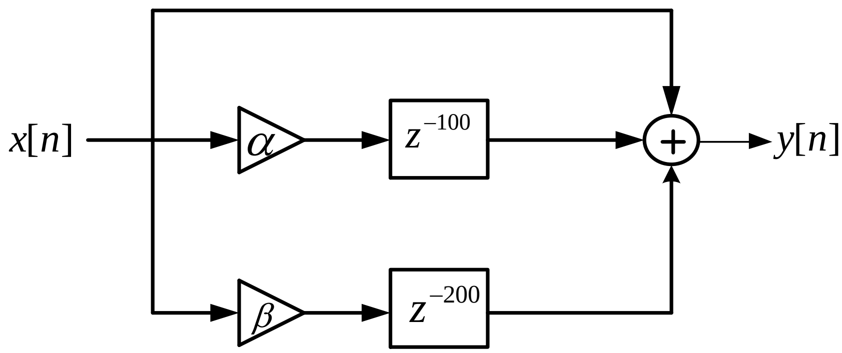

Figure 6: Figure for Problem 6(d)

Problem 7 The Fourier transform of $y_c(t)$ is sketched in Fig. 7,

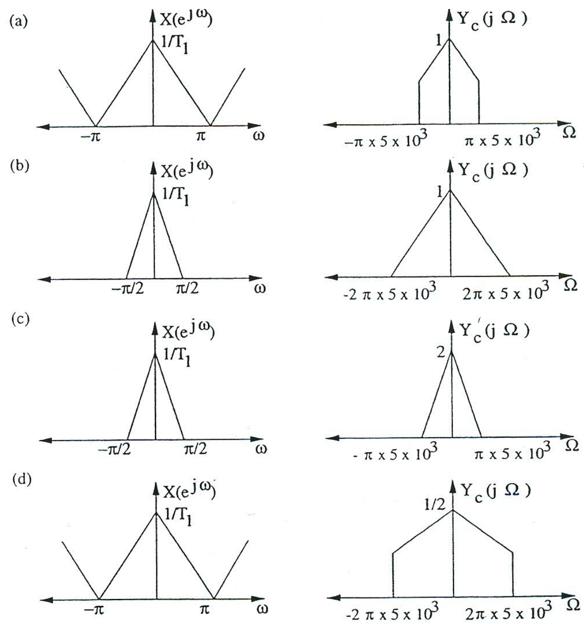

Figure 7: Figure for Problem 7

Problem 8

(a) $H(z)$ is not a polynomial in z or z^{-1} the system is IIR.

(b) $H(z)$ is not a polynomial in z or z^{-1} the system is IIR.

(c) $H(z)$ is a polynomial of z^{-1} the system is an FIR filter.