University of Illinois at Urbana-Champaign

ECE 310: Digital Signal Processing

PROBLEM SET 6: SOLUTIONS

Chandra Radhakrishnan *Peter Kairouz*

Problem 1
From the difference equation:

$$\begin{aligned} y[n] &= h_0 x[n] + h_1 x[n-1] + h_2 x[n-2] \\ \Rightarrow Y(z) &= h_0 X(z) + h_1 z^{-1} X(z) + h_2 z^{-2} X(z) \\ \therefore\ H(z) &= h_0 + h_1 z^{-1} + h_2 z^{-2} \\ H_d(\omega) &= h_0 + h_1 e^{-j\omega} + h_2 e^{-j2\omega} \end{aligned}$$

From the problem, the filter is given as linear-phase FIR. Therefore, the FIR filter must be type-1 or type-2 generalized linear phase. Also, the problem asks you design a bandstop filter ($H_d\left(\frac{5\pi}{6}\right) = 0$). Since an FIR filter with even symmetry (type-1 GLP) and N odd is the only type of FIR filter that can fulfill the bandstop and linear-phase requirements, $h_0 = h_2$ and

$$H_d(\omega) = h_0 + h_1 e^{-j\omega} + h_0 e^{-j2\omega}$$

Plugging in the two conditions that $H_d(0) = 1$ and $H_d\left(\frac{5\pi}{6}\right) = 0$, a system of equations is obtained:

$$\begin{aligned} 1 &= 2h_0 + h_1 \\ 0 &= \frac{3}{2}h_0 - \frac{\sqrt{3}}{2}h_1 + j\left(\frac{\sqrt{3}}{2}h_0 - \frac{1}{2}h_1\right) \end{aligned}$$

This systems of equations is overdetermined (more equations than unknowns) since both the imaginary and real parts of the second equation must be zero. Solving this system of equations, it is obtained that

$$\begin{aligned} h_0 &= 2 - \sqrt{3} \\ h_1 &= 2\sqrt{3} - 3 \\ h_2 &= h_0 = 2 - \sqrt{3} \end{aligned}$$

Problem 2 For any GLP filter, $H_d(\omega) = R(\omega)e^{j(\alpha - M\omega)}$, where $R(\omega)$ is real.

1. $\{h_n\}_{n=0}^{2} = \{2, 1, 1\}$

   Since $h[n]$ has no symmetry, the filter is not GLP.

2. $\{h_n\}_{n=0}^{2} = \{1, 2, 3\}$

   Since $h[n]$ has no symmetry, the filter is not GLP.

3. $\{h_n\}_{n=0}^{2} = \{-1, 3, 1\}$

   The unit-pulse response $h[n]$ is asymmetric but the middle coefficient is nonzero, which prevents $H_d(\omega)$ from being expressed as $H_d(\omega) = R(\omega)e^{j(\alpha - M\omega)}$ where $R(\omega)$ is real. Therefore, the filter is not GLP.

4. $\{h_n\}_{n=0}^{4} = \{1, 1, 1, -1, -1\}$

   Since $h[n]$ has no symmetry, the filter is not GLP.

5. $\{h_n\}_{n=0}^{2} = \{1, 0, -1\}$

The given filter is asymmetric about its midpoint and the middle coefficient ($h[1] = 0$) is zero. Therefore, the filter is a type-2 GLP filter. Hence, $M = \frac{N-1}{2} = 1$. Following the same procedure in (a) to determine $R(\omega)$, which also will determine α,

$$H_d(\omega) = 1 - 2e^{-j2\omega} = e^{-j\omega}(e^{j\omega} - e^{-j\omega}) = e^{-j\omega}(2j\sin(\omega)) = e^{j(\frac{\pi}{2}-\omega)}(2\sin(\omega))$$

Therefore, $R(\omega) = 2\sin(\omega)$, $\alpha = \frac{\pi}{2}$, and M is verified. Taking a look at the phase of $H_d(\omega)$ to determine if the filter is linear-phase:

$$\angle H_d(\omega) = \begin{cases} \frac{\pi}{2} - \omega, & 2\sin(\omega) > 0 \Rightarrow 0 < \omega < \pi \\ \frac{3\pi}{2} - \omega, & 2\sin(\omega) < 0 \Rightarrow -\pi < |\omega| < 0 \end{cases}$$

Since $H_d(\omega)$ has a π jump at $\omega = 0$, the filter is not linear-phase.

6. $\{h_n\}_{n=0}^{3} = \{2, 1, 1, 2\}$

The given filter is symmetric. Therefore, this filter is a type-1 GLP filter. Hence, $\alpha = 0$ and $M = \frac{N-1}{2} = \frac{3}{2}$. Following the same procedure in (a) to determine $R(\omega)$,

$$\begin{aligned} H_d(\omega) &= 2 + e^{-j\omega} + e^{-j2\omega} + 2e^{-j3\omega} \\ &= e^{-j\frac{3\omega}{2}}(2e^{j\frac{3\omega}{2}} + e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}} + 2e^{-j\frac{3\omega}{2}}) \\ &= e^{-j\frac{3\omega}{2}}\left(4\cos\left(\frac{3\omega}{2}\right) + 2\cos\left(\frac{\omega}{2}\right)\right) \end{aligned}$$

Therefore, $R(\omega) = 4\cos\left(\frac{3\omega}{2}\right) + 2\cos\left(\frac{\omega}{2}\right)$, and α and M are verified.
Taking a look at the phase of $H_d(\omega)$ to determine if the filter is linear-phase:

$$\angle H_d(\omega) = \begin{cases} -\frac{3\omega}{2}, & R(\omega) > 0 \\ -\frac{3\omega}{2} + \pi, & R(\omega) < 0 \end{cases}$$

Since $R(\omega)$ changes sign at $\omega = \pm 0.41\pi$, the filter is not linear-phase.

Problem 3

(i) $y[n] = \frac{2}{5}x[n] - x[n-1] + x[n-2] - \frac{2}{5}x[n-3]$
Similarly to (a), the unit pulse response is: $h[n] = \frac{2}{5}\delta[n] - \delta[n-1] + \delta[n-2] - \frac{2}{5}\delta[n-3]$ FIR system.

The unit pulse response $h[n]$ has even length and odd symmetry, thus it has Type II GLP. Further,

$$\begin{aligned} H_d(\omega) &= \frac{2}{5} - e^{-j\omega} + e^{-j2\omega} - \frac{2}{5}e^{-j3\omega} \\ &= \frac{2}{5}e^{-j3\omega/2}\left(e^{j3\omega/2} - e^{-j3\omega/2}\right) - e^{-j3\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right) \\ &= \frac{2}{5}e^{-j3\omega/2}\Big(2j\sin(3\omega/2)\Big) - e^{-j3\omega/2}\Big(2j\sin(\omega/2)\Big) \\ &= e^{-j3\omega/2}\left(\frac{4}{5}j\sin(3\omega/2) - 2j\sin(\omega/2)\right) \\ &= e^{j(\frac{\pi}{2}-3\omega/2)}\left(\frac{4}{5}\sin(3\omega/2) - 2\sin(\omega/2)\right) \end{aligned}$$

From the equation above, $R(\omega) = \frac{4}{5}\sin(3\omega/2) - 2\sin(\omega/2), \alpha = \frac{\pi}{2}, M = \frac{3}{2}$. Since $R(\omega)$ changes sign at $\omega = 0$, the filter does not have linear phase. (In general, filters with antisymmetric coefficients cannot have linear phase.)

(ii) The unit pulse response is: $h[n] = \frac{1}{3}\delta[n] + \delta[n-1] - \frac{3}{4}\delta[n-2]$ FIR system.
There is no symmetry for this $h[n]$, therefore it does not have GLP.

(iii) $y[n] = x[n] + x[n-2] + x[n-4]$

The unit pulse response is: $h[n] = \delta[n] + \delta[n-2] + \delta[n-4]$ FIR system.

Since $h[n]$ has odd length and even symmetry, it has Type I GLP. Also,

$$H_d(\omega) = 1 + e^{-j2\omega} + e^{-j4\omega} \quad H_d(\omega) = e^{-j2\omega}\big(2\cos(2\omega) + 1\big)$$

Therefore, we have $R(\omega) = 2\cos(2\omega) + 1, M = 2, \alpha = 0$.
Since $R(\omega)$ changes sign in the range $-\pi \leq \omega < \pi$, the filter does not have linear phase.

Problem 4

1. The frequency response of the length-N low-pass, generalized linear phase filter with cutoff frequency ω_c of $\frac{\pi}{3}$ is

$$H_d(\omega) = \begin{cases} e^{-j(\frac{N-1}{2})\omega}, & |\omega| \leq \omega_c \\ 0, & \text{otherwise} \end{cases}$$

Let $M = \frac{N-1}{2}$,

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-jM\omega}e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j(n-M)\omega}d\omega \\ &= \frac{1}{2\pi}\frac{1}{j(n-M)}\left(e^{j(n-M)\omega_c} - e^{-j(n-M)\omega_c}\right) \\ &= \frac{1}{\pi(n-M)}\sin\left((n-M)\omega_c\right) \\ &= \frac{\omega_c}{\pi} sinc((n-M)\omega_c) \end{aligned}$$

Since $\omega_c = \frac{\pi}{3}$,

$$\{h_n\}_{n=-\infty}^{\infty} = \frac{1}{3} sinc\left(\frac{\pi}{3}\left(n - \frac{N-1}{2}\right)\right)$$

With the rectangular window, the filter coefficients are

$$h_{\text{rect}}[n] = \begin{cases} \frac{1}{3} sinc\left(\frac{\pi}{3}\left(n - \frac{N-1}{2}\right)\right), & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

2. Replacing the rectangular window with the hamming window,

$$h_{\text{hamming}}[n] = \begin{cases} \left(0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)\right)\frac{1}{3} sinc\left(\frac{\pi}{3}\left(n - \frac{N-1}{2}\right)\right), & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

Problem 5

1. Since the filter has an even number of coefficients, the coefficients need to be antisymmetric in order to realize a high-pass filter. Therefore, the filter is type-2 GLP.

2. One can verify that $H_d(\omega)$ has the same expression for $\frac{3\pi}{4} \leq \omega \leq \pi$ and $\pi \leq \omega \leq \frac{5\pi}{4}$ using the anti-symmetry and the 2π shift in the real part of the frequency response. Therefore,

$$H_d(\omega) = \begin{cases} e^{j(\frac{\pi}{2} - \frac{99}{2}\omega)}, & \frac{3\pi}{4} \leq \omega \leq \frac{5\pi}{4} \\ 0, & \text{otherwise} \end{cases}$$

In order to use the frequency sampling method, the inverse DFT of $H[m]$ is needed,

$$H[m] = \begin{cases} e^{j(\frac{\pi}{2} - \frac{99}{2}\frac{2\pi}{100}m)}, & 38 \le m \le 62 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned}
h[n] &= \frac{1}{100}\sum_{m=38}^{62} e^{j(\frac{\pi}{2} - \frac{99}{2}\frac{2\pi}{100}m)} e^{j\frac{2\pi}{100}mn} \\
&= \frac{e^{j\frac{\pi}{2}}}{100}\sum_{m=38}^{62} e^{j\frac{2\pi}{100}(n-\frac{99}{2})m} \\
&= \frac{e^{j\frac{\pi}{2}}}{100}\sum_{k=0}^{24} e^{j\frac{2\pi}{100}(n-\frac{99}{2})(k+38)} \\
&= \frac{e^{j\frac{\pi}{2}}}{100} e^{j\frac{2\pi}{100}(n-\frac{99}{2})38}\sum_{k=0}^{24} e^{j\frac{2\pi}{100}(n-\frac{99}{2})k} \\
&= \frac{e^{j\frac{\pi}{2}}}{100} e^{j\frac{2\pi}{100}(n-\frac{99}{2})38} \cdot \frac{1 - e^{j\frac{2\pi}{100}(n-\frac{99}{2})25}}{1 - e^{j\frac{2\pi}{100}(n-\frac{99}{2})}} \\
&= \frac{e^{j\frac{\pi}{2}}}{100} e^{j\frac{2\pi}{100}(n-\frac{99}{2})38} \cdot \frac{e^{j\frac{\pi}{100}(n-\frac{99}{2})25}}{e^{j\frac{\pi}{100}(n-\frac{99}{2})}} \cdot \frac{e^{-j\frac{\pi}{100}(n-\frac{99}{2})25} - e^{j\frac{\pi}{100}(n-\frac{99}{2})25}}{e^{-j\frac{\pi}{100}(n-\frac{99}{2})} - e^{j\frac{\pi}{100}(n-\frac{99}{2})}} \\
&= \frac{e^{j\frac{\pi}{2}}}{100} e^{j\frac{\pi}{100}(n-\frac{99}{2})(76+25-1)} \frac{sin(25\frac{\pi}{100}(n-\frac{99}{2}))}{sin(\frac{\pi}{100}(n-\frac{99}{2}))} \\
&= \frac{e^{j\frac{\pi}{2}}}{100} e^{j\pi(n-\frac{99}{2})} \frac{sin(25\frac{\pi}{100}(n-\frac{99}{2}))}{sin(\frac{\pi}{100}(n-\frac{99}{2}))} \\
&= -\frac{(-1)^n}{100} \frac{sin(25\frac{\pi}{100}(n-\frac{99}{2}))}{sin(\frac{\pi}{100}(n-\frac{99}{2}))}
\end{aligned}$$

Problem 6

1. Procedure to determine the 3dB cutoff is as follows:

$$|H(\Omega)|^2 = H_L(j\Omega)H_L^*(j\Omega) = \left(\frac{j\Omega}{j\Omega+2}\right)\left(\frac{-j\Omega}{-j\Omega+2}\right) = \frac{\Omega^2}{\Omega^2+4}$$

$$|H(\Omega_c)|^2 = \frac{1}{2} \;\Rightarrow\; 2\Omega_c^2 = \Omega_c^2 + 4 \;\Rightarrow\; \boxed{\Omega_c = 2\ \frac{\text{rad}}{\text{sec}}}$$

$$\Omega_c = \alpha\tan(\omega_c/2) \;\Rightarrow\; \alpha = \frac{2}{\tan(\pi/6)} = 2\sqrt{3}$$

2.

$$H(z) = H_L(s)\Big|_{s=\alpha\frac{1-z^{-1}}{1+z^{-1}}}$$

$$H(z) = \frac{\alpha\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}{\alpha\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2} = \frac{\alpha(1-z^{-1})}{\alpha(1-z^{-1})+2(1+z^{-1})} = \frac{\alpha-\alpha z^{-1}}{(2+\alpha)+(2-\alpha)z^{-1}}$$

$$H(z) = \frac{\sqrt{3}-\sqrt{3}z^{-1}}{(1+\sqrt{3})+(1-\sqrt{3})z^{-1}}\ ,\ |z| > \frac{1}{2+\sqrt{3}}$$

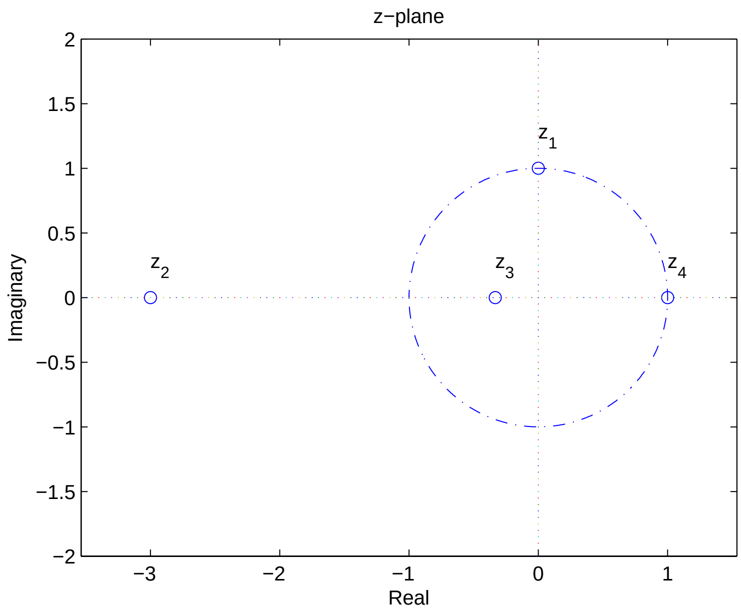
Problem 7

1. $z_1 = \frac{1+s_1}{1-s_1} = \frac{1+j}{1-j} = \frac{(1+j)^2}{(1-j)(1+j)} = j$

2. $z_2 = \frac{1+s_2}{1-s_2} = \frac{1+2}{1-2} = -3$

3. $z_3 = \frac{1+s_3}{1-s_3} = \frac{1+(-2)}{1-(-2)} = -\frac{1}{3}$

4. $z_4 = \frac{1+s_4}{1-s_4} = \frac{1+0}{1-0} = 1$



Note that a point in the left-half plane of the s-plane is mapped to within the unit circle in the z-plane, and a point in the right-half plane of the s-plane is mapped to outside the unit circle in the z-plane. Also, note that the points on the imaginary axis in the s-plane are mapped to the unit circle in the z-plane.

Problem 8

Note that $\alpha = 2$. The mapping between frequency domains is $\Omega = 2\tan(\omega/2)$. Thus,

$$\Omega_c = 2\tan(\omega_c/2) = 2\tan(3\pi/10) = 2.7528 \text{ rad/sec}$$
$$\Omega_s = 2\tan(\omega_s/2) = 2\tan((6\pi-\pi)/20) = 2 \text{ rad/sec}$$
$$\text{Thus, } (\Omega_c - \Omega_s) = 0.7528 \text{ rad/sec}$$

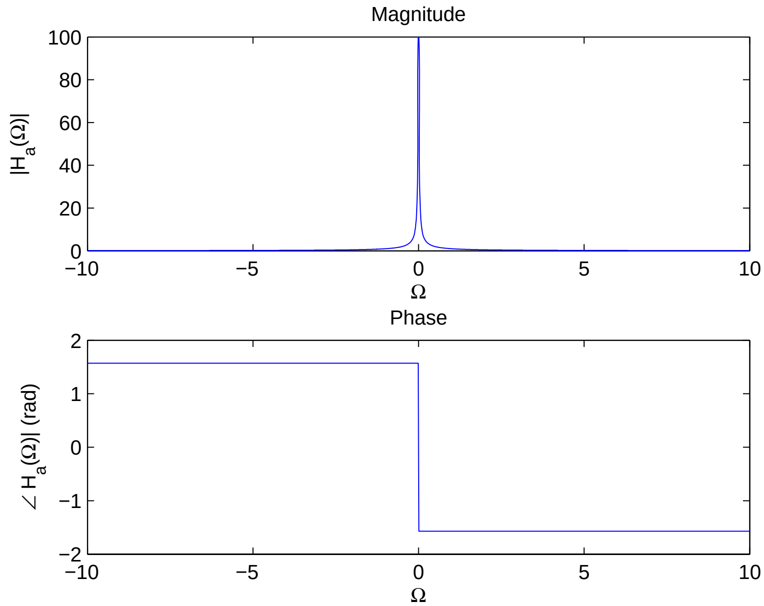
Problem 9

1.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(T/2)(1+z^{-1})}{1-z^{-1}} \Rightarrow Y(z) - z^{-1}Y(z) = (T/2)(X(z) + z^{-1}X(z))$$
$$y[n] = \frac{T}{2}(x[n] + x[n-1]) + y[n-1]$$

2. $H_a(\Omega) = H_L(j\Omega) = \frac{1}{j\Omega}$

$$|H_a(\Omega)| = \frac{1}{|\Omega|} \qquad \angle H_a(\Omega) = \begin{cases} -\pi/2, & \Omega > 0 \\ \pi/2, & \Omega < 0 \end{cases}$$



3.

$$H_d(\omega) = H(z)\big|_{z=e^{j\omega}} = \frac{T}{2}\left(\frac{1+e^{-j\omega}}{1-e^{-j\omega}}\right) = -j\frac{T}{2}\left(\frac{\cos(\omega/2)}{\sin(\omega/2)}\right) = -j\frac{T}{2}\ \text{ctan}\left(\frac{\omega}{2}\right)$$

4. The pole at $z = 1$ implies that inputs are of the form $x[n] = cu[n]$, where c is a non-zero constant, which will lead to an unbounded output. Thus, the input $x[n]$ should contain no term of this form.

5. $H(z)$ has a zero at $z = -1$. Consider an arbitrary point in the s-plane of the form $s_0 = \sigma_0 + j\Omega_0$. With $\alpha = 2/T$, this point is mapped to:

$$z_0 = \frac{\alpha + s_0}{\alpha - s_0} = \frac{\alpha + \sigma_0 + j\Omega_0}{\alpha - \sigma_0 - j\Omega_0}$$

$$\text{where } |z_0| = \left(\frac{(\alpha+\sigma_0)^2 + \Omega_0^2}{(\alpha-\sigma_0)^2 + \Omega_0^2}\right)$$

Thus, the point s_0 from the s-plane is mapped onto the unit circle if and only if $\sigma_0 = 0$. Therefore, the point corresponding to the zero at $z = -1$ must lie on the imaginary axis in the s-plane. Further, note that

$$\lim_{\Omega\to\infty}\frac{\alpha+\Omega}{\alpha-\Omega} = \lim_{\Omega\to-\infty}\frac{\alpha+\Omega}{\alpha-\Omega} = -1$$

Thus, the zero in $H(z)$ at $z = -1$ corresponds to $\boxed{\Omega = \pm\infty}$.

The solutions to problems 10 - 12 are sketched below,

(10) (a) For $L = 1$

Transition bandwidth: $\frac{2(\pi - \omega_c)}{T_1}$

here $\omega_c = 0.9070\pi$

$\Rightarrow$ Transition bandwidth $= 2(\pi - 0.9070\pi) \times 44100$

$= 2 \times \pi \times 4.1 \times 10^3$ rads

$\Rightarrow T_{band} = 4.1$ KHz

(b) $L = 2$

Transition bandwidth $= \frac{2\pi}{T_2} - \frac{2\omega_c}{T_1}$

$= \frac{2\pi L}{T_1} - \frac{2\omega_c}{T_1}$

$= 2\pi \times 2 \times 44100 - 2 \times 0.9070\pi \times 44100$

$= 2\pi \times 48.2 \times 10^3$

$\Rightarrow T_{band} = 48.2$ KHz

(c) $L = 4$

Transition band $= \frac{2\pi \times 4}{T_1} - \frac{2 \times \omega_c}{T_1}$

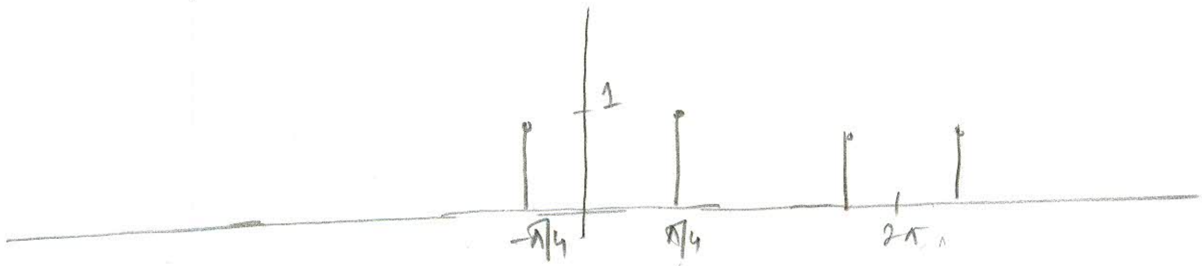
$= 2\pi \times 4 \times 44100 - 2 \times 0.9070 \times \pi \times 44100$

$= 2\pi \times 136.4 \times 10^3$

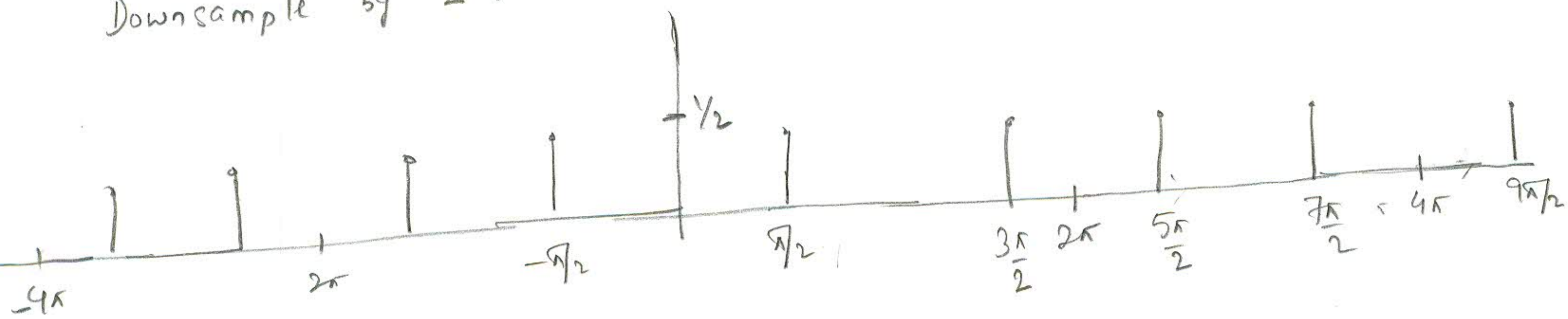
$\Rightarrow T_{band} = 136.4$ KHz

(11) a) $x(n) = \cos\left(\frac{\pi}{4} n\right)$

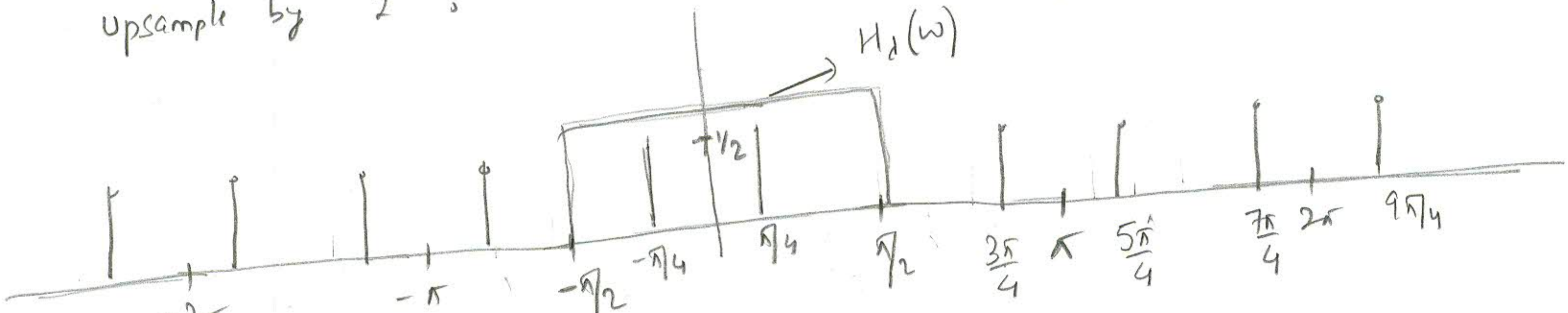
The spectrum of $X_d(\omega)$ is shown below,

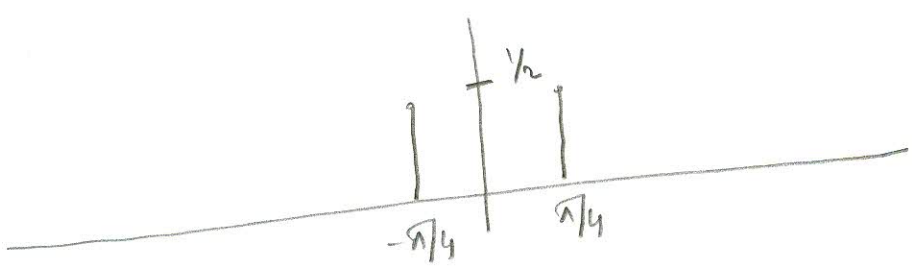


Downsample by 2 :



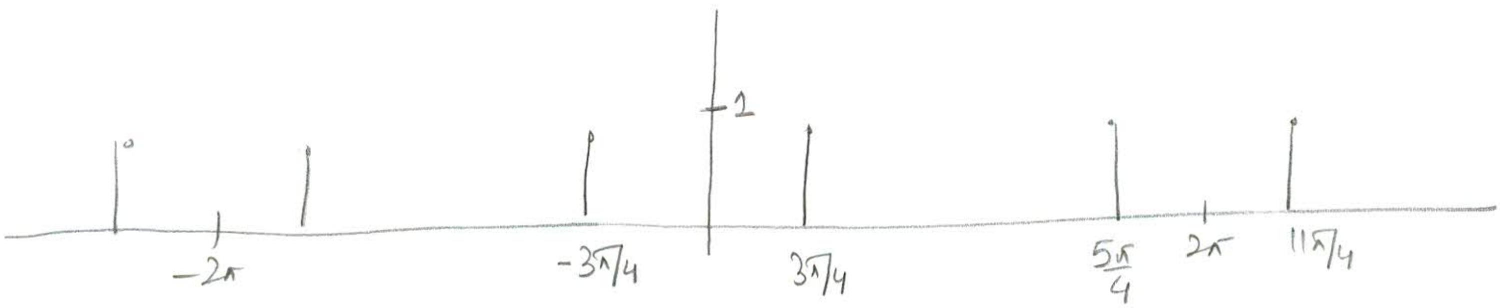
upsample by 2 ; and low pass filtering



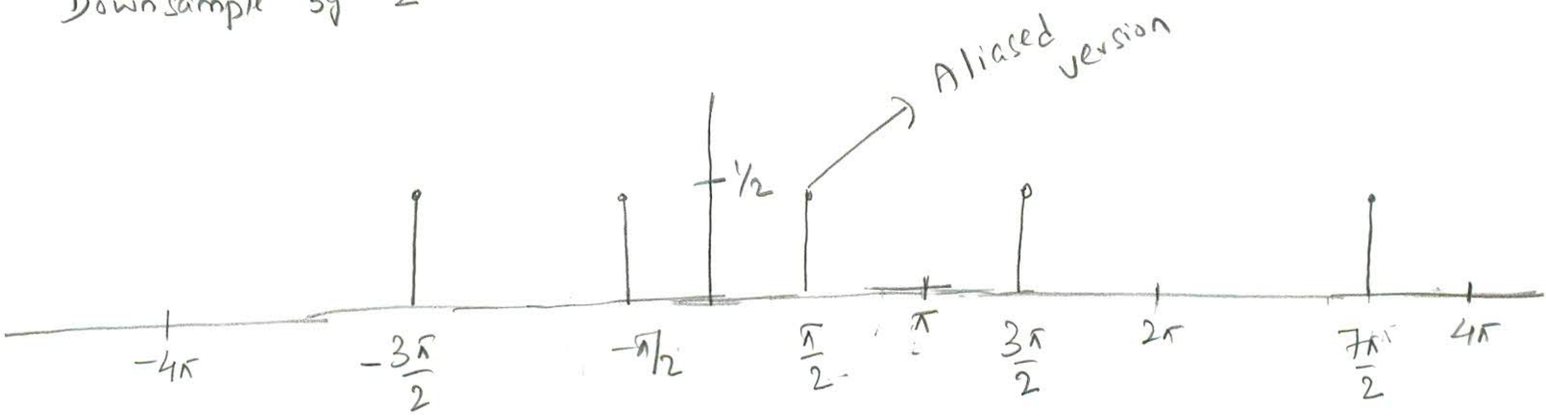


hence $y(n) = \frac{1}{2} \cos \frac{\pi}{4} n$

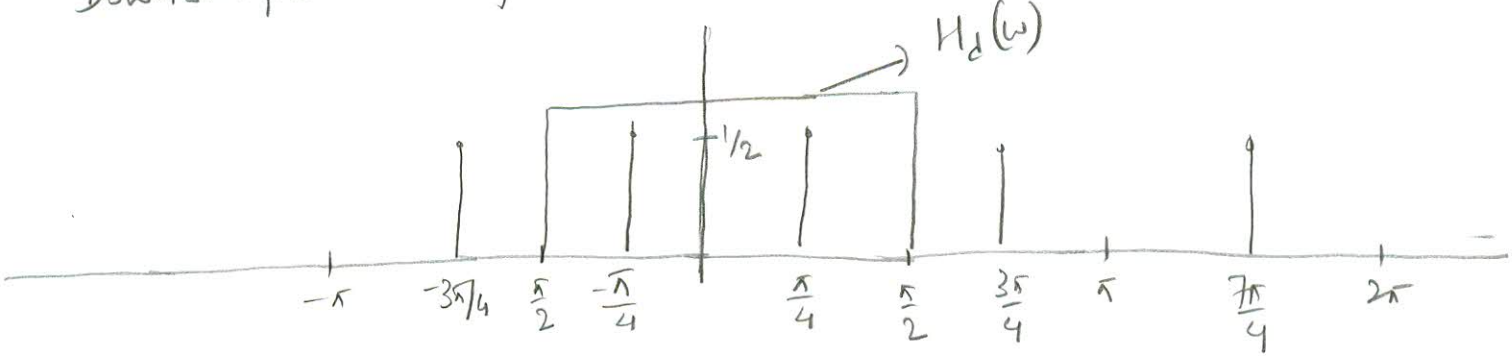
(b) $x[n] = \cos \frac{3\pi}{4} n$



Downsample by 2



Downsample and filter :

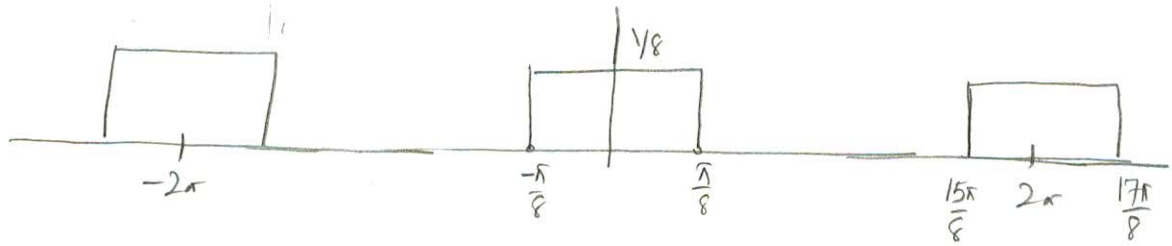


Note we get aliased version

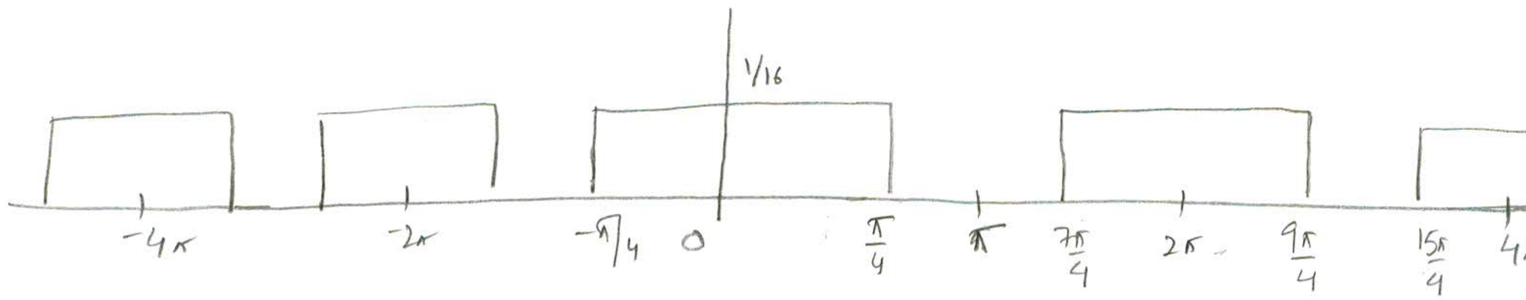
$$y[n] = \frac{1}{2} \cos \frac{\pi}{4} n$$

(c) $x(n) = \dfrac{\sin\left(\frac{\pi n}{8}\right)}{\pi n}$

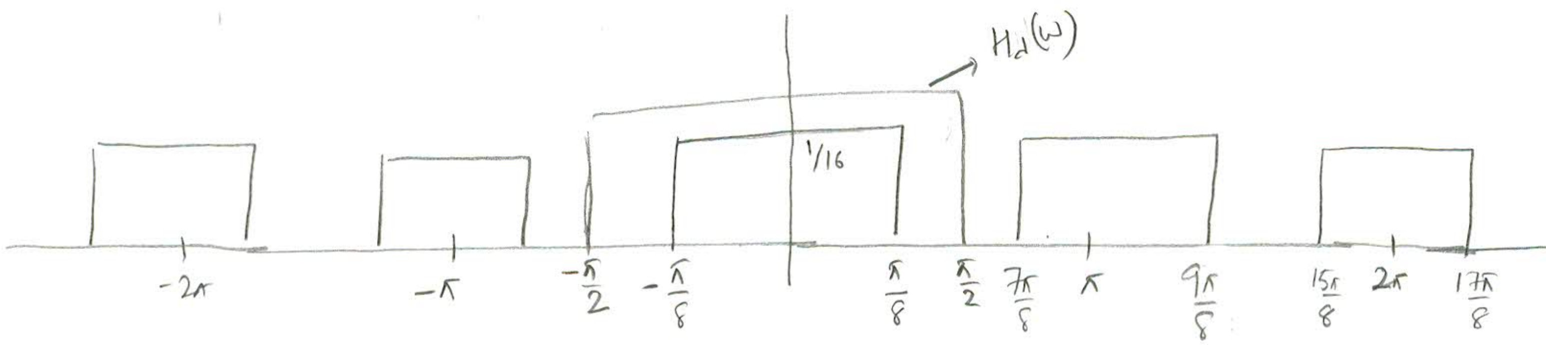
$\Rightarrow$



down sample by 2,

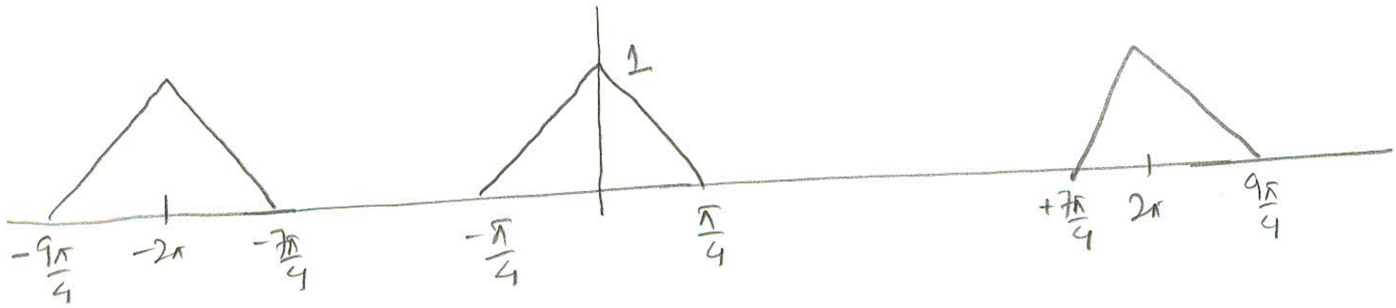


Up sample by 2 & filter:

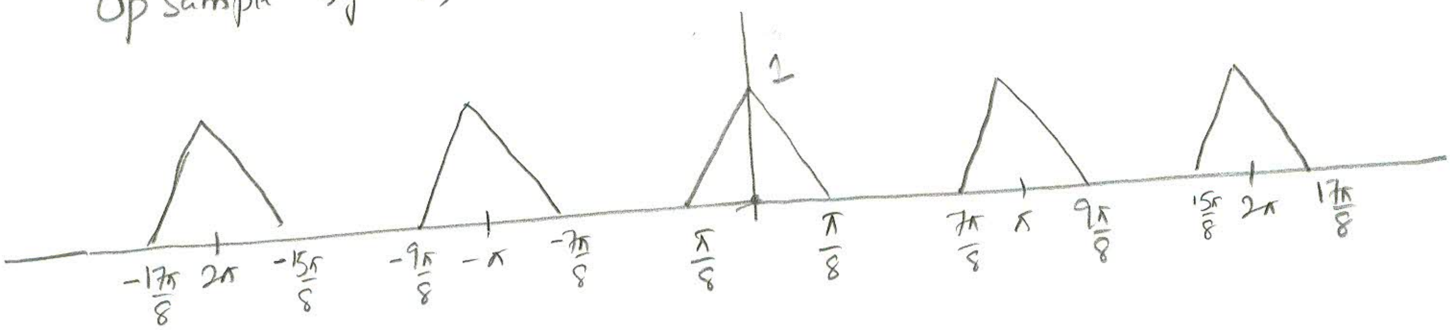


$\Rightarrow \quad y(n) = \dfrac{1}{2} \dfrac{\sin\left(\frac{\pi n}{8}\right)}{\pi n}$

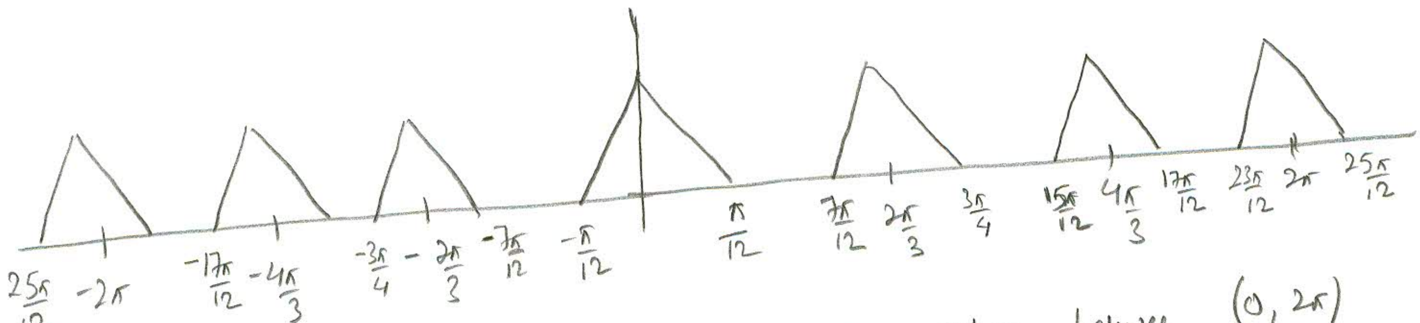
(12) (a) The spectrum of $X_d(\omega)$ is shown below.



up sample by 2,



(b)



Note there are $L-1 = 2$ copies between $(0, 2\pi)$

(c)