

University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 311: Digital Signal Processing Lab
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LAB 3: Sampling, Convolution, LTI systems, and
Difference Equations
Summer 2011

1 Overview

In this lab we will study sampling, convolution, and analyze LTI systems using MATLAB. The objective of this lab is to teach you the effects of sampling on continuous time signals and their spectra. We will also take a deeper look at convolution and properties of LTI systems.

2 Sampling

The sampling theorem specifies conditions under which a bandlimited continuous-time signal can be completely represented by discrete samples. The resulting discrete-time signal $x[n] = x_c(nT)$ contains all the information in the continuous-time signal so long as the continuous-time signal is sufficiently bandlimited in frequency, i.e., $X_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T$. When this condition is satisfied, the original continuous-time signal can be perfectly reconstructed by interpolating between samples of $x[n]$.

Consider the sinusoidal signal,

$$x(t) = \sin(\Omega_0 t)$$

If $x(t)$ is sampled with frequency $\Omega_s = 2\pi/T$ rad/sec, then the discrete-time signal $x[n] = x(nT)$ is equal to

$$x[n] = \sin(\Omega_0 nT)$$

Assume sampling frequency is fixed at $\Omega_s = 2\pi(8192)$ rad/sec.

1. Assume $\Omega_0 = 2\pi(1000)$ rad/sec and define $T = 1/8192$. Create the vector $\mathbf{n}=[0:8191]$, so that $\mathbf{t}=\mathbf{n}*T$ contains 8192 samples in the interval $0 \leq t \leq 1$. Create a vector which contains the samples of $x(t)$.
2. Display first 50 samples of $x[n]$ versus n using `stem`. Display the first fifty samples of $x(t)$ versus the sampling time using `plot`.

Note: `plot` displays a continuous-time signal given the samples in \mathbf{x} .

Now consider the following signal:

$$x(t) = a_1 \cos(\Omega_0 t) + b_1 \cos(\Omega_1 t)$$

We can sample this signal at a sampling rate Ω_s to yield a discrete-time signal $x[n]$. We can now use the DFT (say length M) to compute the spectrum of this signal. Note the DFT samples $X[k]$ are related to the analog frequency by

$\Omega = (2\pi k/M)/T$ where T is the sampling interval. Also the spectrum of this signal will have two periodic sines. We have seen earlier that to distinguish two frequencies we must have (assuming 50% overlap),

$$MT \geq \frac{2\pi}{|\Omega_1 - \Omega_0|}.$$

The larger M gives higher peaks and smaller M gives wider sidelobes. The sampling interval T must be chosen correctly otherwise one has aliasing where the higher frequency can appear as a lower frequency.

3 Linear Time-Invariant Systems

In discrete time, linearity provides the ability to completely characterize a system in terms of its response $h_k[n]$ to signals of the form $\delta[n - k]$ for all k . If a linear system is also time-invariant, then the responses $h_k[n]$ satisfy $h_k[n] = h[n - k]$. The combination of linearity and time-invariance therefore allows a system to be completely described by its impulse response $h[n]$ since the output of the system $y[n]$ is related to the input $x[n]$ through the convolution sum.

$$y[n] = \sum_{n=-\infty}^{n=\infty} h[n - m]x[m] \quad (1)$$

The MATLAB function `conv` computes the convolution sum in (1) assuming that $x[n]$ and $h[n]$ are finite length sequences. If $x[n]$ is non zero only on the interval $n_x \leq n \leq n_x + N_x - 1$ and $h[n]$ is non-zero only on the interval $n_h \leq n \leq n_h + N_h - 1$, then $y[n]$ is non-zero in $n_x + n_h \leq n \leq n_x + n_h + N_x + N_h - 2$ meaning that `conv` computes $N_x + N_h - 1$ samples. Consider the finite length signal

$$x[n] = \begin{cases} 1, & 0 \leq n \leq 5 \\ 0, & \text{otherwise,} \end{cases}$$

Use `conv` to compute $y[n] = x[n] * x[n]$. Store the result in `y`. Plot `y` using `stem` command. Note `conv` does not return an index vector. Index vector `ny` must be constructed such that `ny(i) = y[ny(i)]`. `ny(1)` must be $n_x + n_y$.

A LTI system can also be represented as a difference equation as shown below:

$$\sum_{k=0}^K a_k y[n - k] = \sum_{m=0}^M b_m x[n - m] \quad (2)$$

where $x[n]$ is the system input and $y[n]$ is the system output. If `x` is a MATLAB vector containing the input $x[n]$ on the interval $n_x \leq n \leq n_x + N_x - 1$ and the vectors `a` and `b` contain the coefficients a_k and b_m then `filter(b,a,x)` returns the output of the causal LTI system satisfying

$$\sum_{k=0}^K a[k + 1]y[n - k] = \sum_{m=0}^M b[m + 1]x[n - m]$$

Note that $a[k + 1] = a_k$ and $b[m + 1] = b_m$, since MATLAB requires all vectors to begin at one. For example, to specify the system described by the difference equation

$$y[n] + 2y[n - 1] = x[n] - 3x[n - 1],$$

you would define `a = [1 2]` and `b = [1 -3]`.

Also note that the output of `filter` contains samples of $y[n]$ in the same interval as the samples of `x`. Also, `filter` need samples in the interval $n_x - M \leq n \leq n_x - 1$ in order to compute the first output sample of $y[n]$. If they are not provided, `filter` assumes these samples are zero.

Define `a1` and `b1` to describe the causal LTI system shown below:

$$y[n] = 0.5x[n] + x[n - 1] + 2x[n - 2]$$

```
x = [1 2 3 4];
a1 = [1];
b1 = [0.5 1 2];
y1 = filter(b1,a1,x)
```

Use `filter` to compute response $y[n]$ to input $x[n] = nu[n]$: Try the following system:

$$y[n] = 0.8y[n - 1] + 2x[n]$$

```
x = [1 2 3 4];
a1 = [1 -0.8];
b1 = [2];
y1 = filter(b1,a1,x)
```

4 Homework - Due 07/05/2011 at 5:00 PM

1. Consider a continuous sinusoidal signal $x_a(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t)$, where $f_1 = 1kHz$ and $f_2 = 1.4kHz$. This signal is sampled at $\Omega_s = 2\pi 10^4$ rad/s.

(a) What is the discrete sequence $x[n]$?

(b) Generate $N = 5000$ samples of $x[n]$ and compute the N -point DFT $X[k]$ of $x[n]$, $n = 0, 1, \dots, N - 1$ in MATLAB. Plot the magnitude of sampled version of $X_a(\Omega)$ using the DFT sequence $X[k]$. Keep in mind that the magnitude is in dB-scale and x-scale should have a physical meaning (Hz). Can you identify f_1 and f_2 from the plot? Hint: use the following routine:

```
% T: sampling period
% x: input sequence
XF = fft(x,N);
XF = [XF(N/2+1:N) XF(1:N/2)];
fq = [-N/2:N/2-1]/N*(1/T);
plot(fq,10*log10(abs(XF)));
axis([-1/T/2 1/T/2 -inf inf]);
xlabel('Frequency (Hz)');
ylabel('Magnitude (dB)');
grid;
```

(c) Repeat the same steps as in (b) for $N = 100$ and $N = 20$.

(d) Provide the theoretical value of N over which the main-lobes of two cosines do not overlap. Does this match with your empirical results in (a) and (b)?

2. Let $x_a(t) = \cos(2\pi t) + 0.8\cos((14/15)\pi t)$. In this exercise, you will investigate how the choice for the number of samples of $x_a(t)$, N , and the sampling period, T , affect the analysis of the spectrum. A 256-point DFT will be used to get fine sampling of $X_d(\omega)$ for each choice of N and T . In order to conveniently vary N and T , create the file `test.m` to generate N samples of $x_a(t)$ at intervals of T . The following code can be used to serve this purpose:

```
function x = test(N,T)
x = zeros(1,256);
n = 0:N-1;
x(1:N) = cos(2*pi*(n)*T) + 0.8*cos((14/15)*pi*(n)*T);
```

Note: the returned vector is zero-padded to length 256, so that a 256-point DFT can always be used. Given this new function, the magnitude of the 256-point DFT for $N = 64$ and $T = 1/30$ can be obtained by typing:

```
x = test(64, 1/30);
X = fft(x);
k = 0:255;
plot(k, abs(X)), xlim([-2 257]);
title('N=64, T=1/30'); xlabel('k'); ylabel(' | X(k) |');
```

(a) Compute and plot the magnitude (in MATLAB) the 256-point DFT $X[k]_{k=0}^{255}$ for each of the following cases:

- | | |
|---------------------|------------------------|
| (i) $N=64, T=1/240$ | (ii) $N=64, T=1/30$ |
| (iii) $N=64, T=1/5$ | (iv) $N=32, T = 1/120$ |
| (v) $N=32, T=1/30$ | (vi) $N=32, T=1/15$ |

(b) For each of the above cases, determine the analog frequencies corresponding to $X[73]$ and $X[173]$.

(c) Use the expression for the DFT of a single truncated sinusoid to explain the effect of the number of samples N and the sampling interval T on the resulting plots.

(d) Given a 2 second long segment of $x_a(t)$, how would you choose the sampling interval T to avoid problems that occur from not sampling at or above the Nyquist rate. The problems that occur are called aliasing.

(e) Given $x_a(t)$ for $-\infty < t < \infty$, and that only $N = 128$ samples are to be acquired, how would you choose T to best resolve the sinusoidal components?

(f) Given that $T = 1/30$, use the previous reasoning to give an estimate of the minimum number N_{min} of samples required to resolve the sinusoids. Also determine N_{min} experimentally (i.e., using test $(N, 1/30)$; how small can N be before you cannot tell that there are two sinusoids?). Demonstrate the experimentally determined N_{min} using plots generated by test $(N, 1/30)$ for $N = N_{min} - 1, N_{min}$, and $N_{min} + 1$.

3. Consider the following $x[n]$ and $h[n]$:

$$x[n] = \begin{cases} 1, & 0 \leq n \leq 5 \\ 0, & \text{otherwise,} \end{cases}$$

$$h[n] = \begin{cases} n, & 0 \leq n \leq 5 \\ 0, & \text{otherwise,} \end{cases}$$

(a) Analytically compute $y[n] = h[n] * x[n]$.

(b) Compute the convolution of $x[n]$ and $h[n]$ using `conv` command. Store the result in `y` and plot `y` using `stem`.

(c) Now compute the convolution $y[n] = h[n+5] * x[n]$. Store the result in `y1` and plot `y1` using `stem`. How does this result compare with the result in part (b).

4. Consider two causal systems defined by the following linear difference equations:

$$\text{System 1: } y_1[n] = \frac{3}{5}y_1[n-1] + x[n]$$

$$\text{System 2: } y_2[n] = \left(\frac{3}{5}\right)^n y_2[n-1] + x[n].$$

Each system satisfies initial rest conditions, which state that if $x[n] = 0$ for $n \leq n_0$ then $y[n] = 0$ for $n \leq n_0$. Define $h_1[n]$ and $h_2[n]$ to be the responses of the System 1 and 2, respectively, to the signal $\delta[n]$.

- (a) Calculate $h_1[n]$ and $h_2[n]$ on the interval $0 \leq n \leq 19$, and store these responses in **h1** and **h2**. Plot each response using **stem**. *Hint:* The filter function can be used to calculate **h1**. However, System 2 is described by a difference equation with non-constant coefficients; therefore, you must either determine **h2** analytically or use a **for** loop rather than **filter** to calculate **h2**.
- (b) For each system, calculate the unit step response on the interval $0 \leq n \leq 19$, and store the response in **s1** and **s2**. Again, **filter** can be used only to calculate the step response of System 1. Use **for** loop to calculate **s2**.
- (c) Note that $h_1[n]$ and $h_2[n]$ are zero for $n \geq 20$ for all practical purposes. Thus **h1** and **h2** contain all we need to know about the response of each system to the unit impulse. Define $z_1[n] = h_1[n]*u[n]$ and $z_2[n] = h_2[n]*u[n]$, where $u[n]$ is the unit step function. Use **conv** to calculate $z_1[n]$ and $z_2[n]$ and store it in vectors **z1** and **z2**. You must first define a vector containing $u[n]$ over an appropriate interval, and then select the subset of the samples produced by **conv(h1,u)** and **conv(h2,u)** that represent the interval $0 \leq n \leq 19$. Since you have truncated two infinite-length signals, only a portion of the outputs of **conv** will contain valid sequence values.
- (d) Plot **s1** and **z1** on the same set of axes. If the two signals are identical, explain why you could have anticipated this similarity. Otherwise, explain any difference between the two signals. On a different set, plot **s2** and **z2**. Again, explain how you might have anticipated any differences or similarities between these two signals.

Deliverables

- Email your code, figures, calculation and answers as a *.pdf* or *.doc* file to **ece311lab.uiuc@gmail.com**. Be sure to name your document in the form- **ECE311Lab3_firstname_lastname.doc/pdf**.
- Late reports will reduce the grade by 20% per day.
- Make sure to present a clear and concise report having figures labeled and centered.
- **Reminder: Homework is due on 07/05/2011**