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LAB 5: **Frequency Response of LSI systems**

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# 1 Overview

In this lab we will use MATLAB to study the frequency response of LSI systems. We will look at the relationship between pole-zero locations and frequency response of a LSI system.

# 2 Frequency Response

Response of any relaxed LTI/LSI system to an arbitrary input signal $x[n]$, is given by the convolution sum formula:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \tag{1}$$

In this input-output relationship,the system is characterized in the time domain by its unit impulse response ($h[n], -\infty \leq n \leq \infty$).
To develop a frequency-domain characterization of the system, we excite the system with the complex exponential

$$x[n] = Ae^{j\omega_0 n}, \quad -\infty < n < \infty \tag{2}$$

where $A$ is the amplitude and $\omega_0$ is an arbitrary frequency confined to the frequency interval $[-\pi, \pi]$. By substituting (1) into (2), we obtain the response

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{k=\infty} h[k] \left[ Ae^{j\omega_0 (n-k)} \right] \\ &= A \left[ \sum_{k=-\infty}^{k=\infty} h[k]e^{-j\omega_0 k} \right] e^{j\omega_0 n} \end{aligned} \tag{3}$$

The term in the brackets in (3) is a function of the frequency variable $\omega_0$ and is the Fourier transform of the unit impulse response $h[k]$ of the system. Hence we denote this function as,

$$H(\omega_0) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k}$$

Note that the function $H(\omega)$ exists if the system is BIBO stable,

$$H(\omega) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

The response of the system to the complex exponential $x[n] = Ae^{j\omega_0 n}$ is given by,

$$y[n] = AH(\omega_0)e^{j\omega_0 n}$$

The response is also a complex exponential with the same frequency as the input and is scaled by a constant factor $H(\omega_0)$. In general, $H(\omega_0)$ is a complex-valued function of the variable $\omega_0$ and can be written as,

$$H(\omega_0) = |H(\omega_0)|e^{\angle H(\omega_0)}$$

The system response to a complex input can now be written as,

$$y[n] = A|H(\omega_0)|e^{j\omega_0 n}e^{j\angle H(\omega_0)}.$$

Note that $|H(\omega_0)|$ and $\angle H(\omega_0)$ completely characterize the effect of the system on exponential input signal of any arbitrary frequency. Since $H(\omega_0)$ determines the response of the system in the frequency domain, it is called the frequency response and the quantities $|H(\omega_0)|$ and $\angle H(\omega_0)$ are respectively called the magnitude and phase response of the system.

## 3 Geometric Interpretation of the Discrete-Time Frequency Response

Recall that the system function of an LTI system can be obtained by taking the $z$- transform of the unit impulse response of the system $h[n]$. The system function can be factored in the form,

$$H(z) = A\frac{\prod_{k=1}^{K}(z - z_k)}{\prod_{m=1}^{M}(z - p_m)}, \tag{4}$$

where the $z_k$ are the $K$ zeros and the $p_m$ are the $M$ poles. The contribution of each pole and each zero to $|H(\omega)|$ depends on the length of the vector from the pole or zero to the point $e^{j\omega}$. Taking the magnitude of (4) and evaluating it at $z = e^{j\omega}$ yields,

$$H(z) = |A|\frac{\prod_{k=1}^{K}|e^{j\omega} - z_k|}{\prod_{m=1}^{M}|e^{j\omega} - p_m|}. \tag{5}$$

Thus the overall magnitude of the frequency response is the magnitude of the constant $A$ times the product of the lengths of the zero vectors divided by the product of the lengths of the pole vectors. Similarly, the contribution of each pole or zero to the phase of the frequency response ($\angle H(\omega)$) is angle formed by the real axis and the vector between the pole or zero and the point $e^{j\omega}$. Taking the phase of (5) we have,

$$\angle H_d(\omega) = \angle A + \sum_{k=1}^{K}\angle(e^{j\omega} - z_k) - \sum_{m=1}^{M}\angle(e^{j\omega} - p_m).$$

From this, the total phase is the phase of the constant $A$ plus the sum of the angle contributions from the zeros minus the sum of the angle contributions from the poles.

# 4 Homework - Due 07/19/2011 at 5:00 PM

1. Consider the simple second-order discrete-time system whose system function is

$$H_1(z) = \frac{1}{1 - 0.9z^{-1} + 0.81z^{-2}}, \quad |z| > 0.9$$

   (a) Define `b1` and `a1` to contain the coefficients of the numerator and denominator polynomials of $H_1(z)$ in the format required by `filter`. Find the poles and zeros for $H_1(z)$.

   (b) Define `omega = [0:511]*pi/256` and `unitcirc = exp(j*omega)` to get the 512 equally spaced points on the unit circle where you will evaluate the frequency response $H_1(\omega)$. Define `polevectors1` to be a $2 \times 512$ matrix where each row contains the complex numbers that result from subtracting one of the pole locations from the corresponding column of `unitcirc`. If `ps1` is a column vector containing the pole locations, you can do this using

```
>> polevectors1 = ones(2,1)*unitcirc - ps1*ones(1,512);
```

   Use `abs` and `atan2` to define `polelength1` and `poleangle1` as the magnitude and angle of each element of `polevectors1`.

   (c) Define `zerovectors1` analogously to `polevectors1` so that it is the $2 \times 512$ matrix containing the vectors from zero locations to the elements of `unitcirc`. Define `zerolength1` and `zeroangle1` to be the magnitude and the phase for these vectors, respectively.

   (d) Plot `polelength1` and `zerolength1` against `omega`. Based on these plots, where do you expect $|H_1(\omega)|$ to have its maxima and minima?

   (e) Use `polelength1` and `zerolength1` to compute $|H_1(\omega)|$ and store the result in `geomH1mag`. Use `poleangle1` and `zeroangle1` to compute $\angle H_1(\omega)$ and store the result in `geomH1phase` (you may find the functions `prod` and `sum` useful). Plot the geometrically derived magnitude and phase, and compare them with those you obtain by computing:

```
>>H1 = freqz(b1,a1,512,'whole');
```

2. Consider the following transfer function,

$$H_1(z) = \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}}, \quad |z| > 0.9$$

   (a) Find and plot the poles and zeros for $H_2(z)$. How do you expect the `polevectors2` or `zerovectors2` for this system to be different than they were for $H_1(z)$?

   (b) Compute the `polevectors2` and `zerovectors2` for $H_2(z)$, as well as the magnitudes and angles for all the vectors. Plot the magnitudes and angles against `omega`. Was your prediction in Part (a) correct?

   (c) Based on changes to the zeros, predict how $H_2(\omega)$ will differ from $H_1(\omega)$. Compute and plot `H2` using `freqz` to confirm your answer.

3. Consider the following transfer function,

$$H_3(z) = \frac{0.25 - (\sqrt{3}/2)z^{-1} + z^{-2}}{1 - (\sqrt{3}/2)z^{-1} + 0.25z^{-2}}, |z| > 0.5$$

   (a) Find and plot the poles and zeros of $H_3(z)$. How are the poles and zero locations related?

(b) Define `polevectors3` and `zerovectors3` analogously to the way you did in parts (a) and (b). Define `polelength3` and `zerolength3` to be the magnitudes of these complex numbers. Plot all of these magnitudes, i.e., the magnitude of each row of `polelength3` and `zerolength3` on same set of axes. How are these magnitudes related? Based on this, how do you expect the frequency response magnitude $|H_3(\omega)|$ to vary with frequency? Use the lengths to compute the frequency response magnitude and store it in `geomH3mag`. Plot `geomH3mag` against `omega`.

(c) Compute `H3` using `freqz` and confirm your answer.

4. For a LTI system described by the difference equation,

$$y[n] = \sum_{m=0}^{M} b_m x[n-M] - \sum_{l=0}^{N} a_l y[n-l]$$

the frequency-response function is given by,

$$H(\omega) = \frac{\sum_{m=0}^{M} b_m e^{-j\omega m}}{1 + \sum_{l=1}^{N} a_l e^{-j\omega l}}$$

Write a MATLAB function `freqresp` to implement this relation. The format should be

```
function [H ] = freqresp(b,a,w)
% frequency response function
% [H] = freqresp(b,a,w)
% b = numerator coefficients
% a = denominator coefficients a(1)=1
% w = frequency location vector.
```

Use the function to compute and plot the magnitude and phase response of the following transfer functions,

$$\begin{aligned} H_1(z) &= 1 + z^{-1} \\ H_2(z) &= 1 - z^{-1} \end{aligned}$$

## Deliverables

- Email your code, figures, calculation and answers as a *.pdf* or *.doc* file to **ece311lab.uiuc@gmail.com**. Be sure to name your document in the form- **ECE311Lab5_firstname_lastname.doc/pdf**.
- Late reports will reduce the grade by 20% per day.
- Make sure to present a clear and concise report having figures labeled and centered.
- **Reminder: Homework is due on 07/19/2011**