Appendix A

Appendix on complex numbers:

A.1 complex numbers

We begin with a review of several properties of complex numbers, their representation, and some of their basic properties. The use of complex numbers, complex-valued functions, and functions of a complex variable will prove essential for an understanding of the material in this text, so it is important that before proceeding with the rest of this material, some basic notions are well understood. Without the ability to manipulate complex numbers and functions, our treatment of discrete-time system theory would be much more cumbersome.

There are many ways in which complex numbers may be represented. Two representations that will be used extensively in this text are the "rectangular" form and the "polar" form. The rectangular form, also called the "Cartesian" form, represents a complex number z as an ordered pair of real numbers, usually written

$$z = x + jy$$

where x and y are real numbers, with x referred to as the "real part" of z and y referred to as the "imaginary part" of z and $j = \sqrt{-1}$. We can write

$$x = \Re z \text{ and } y = \Im z$$

to illustrate taking the "real part" and the "imaginary part" of the complex number z. In polar form, we can write

$$z = re^{j\theta},$$

where $r > 0$ is referred to as the magnitude of the complex number z and θ is the phase or angle of z. We can then express these relationships as

$$r = |z|, \text{ and } \theta = \angle z,$$

and use Euler's relation

$$e^{j\theta} = \cos(\theta) + j\sin(\theta),$$

to relate the complex cartesian and polar representations as

$$r = \sqrt{x^2 + y^2}, \text{ and } \theta = \arctan(y/x).$$

These relationships can be obtained by considering the real and imaginary parts of a complex number as points in the complex (x, y) plane. Then the complex number can be thought of as a vector in the plane from the origin to the point (x, y), with the magnitude of the vector being r and the angle formed from the real line to the vector yielding θ, as in figure A.1.

We see that by Euler's relation, we have

$$z = re^{j\theta} = (r\cos(\theta)) + j(r\sin(\theta))$$

and then we obtain

$$x = r\cos(\theta) \text{ and } y = r\sin(\theta).$$

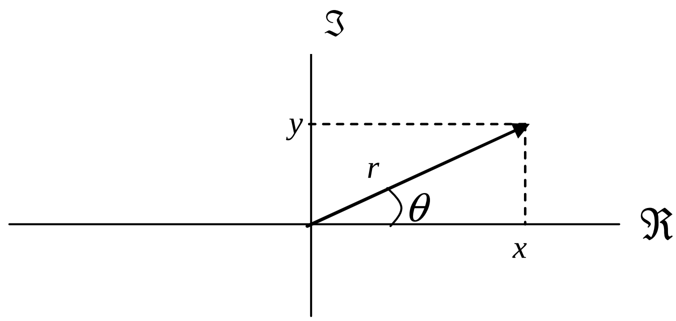


Figure A.1: Vector representation of a complex number, relating the polar and Cartesian forms. Euler's relation can be used to relate the real part and imaginary parts with the magnitude and phase.

We can similarly write

$$\sqrt{x^2 + y^2} = \sqrt{r^2(\cos(\theta)^2 + \sin(\theta)^2)} = r,$$

and

$$\frac{y}{x} = \frac{r\sin(\theta)}{r\cos(\theta)} = \tan(\theta),$$

such that

$$\theta = \arctan(y/x).$$

Complex numbers are simply a useful tool that enables us to describe a wider class of equations than do the real numbers alone. For example, if we consider the equation

$$x^2 + 1 = 0,$$

and ask for what values of x does this equation have a solution? We find that when x takes on values from the real numbers, then there are no solutions. However we can learn more about this equation, and about equations involving higher order polynomials in x if we can introduce a solution to this equation. In order to do so, we must now think of the function

$$f(x) = x^2 + 1$$

not as a function over the reals, but rather as a function over a new number system - one for which a solution to this equation exists. By creating this solution, and giving it the name j, we "create" a number that did not exist in the reals, namely the square root of -1. By constructing a field of numbers over which our algebraic structures behave as we have come to expect, based on the real numbers, we need to introduce a way to add, subtract, and multiply complex numbers. In Cartesian form, we have that the sum of two complex numbers can be written

$$(x_1 + jy_1) + (x_2 + jy_2) = (x_1 + x_2) + j(y_1 + y_2)$$

that is, to add two complex numbers, we simply can use real addition of the real parts and real addition of the imaginary parts, separately to construct the real part and the imaginary part of the sum.

Multiplication of two complex numbers in Cartesian form can be written

$$(x_1 + jy_1)(x_2 + jy_2) = (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1),$$

where we can apply the distributive law of multiplication over addition, and use our newly formed relation that $j^2 = -1$. By using this relation, we can think of j as a variable, and apply the same algebraic manipulations to complex numbers as we would to polynomials in the variable j, so long as once the algebraic manipulations are completed, we agree to replace j^2 with the number -1, and repeat this process until we have exhausted all powers of j. We can then write the resulting complex number in Cartesian form by collecting the remaining terms without this variable, and call them the real part and those that contain the variable become the imaginary part.

Suppose we use a different notation for a complex number, such that

$$c = a + ☺b$$

Using this notation, the real number sitting next to the smiling face is the imaginary part of c and the real number not sitting next to the smiling face is the real part of c. Now, pretend that c is a polynomial in the "variable" ☺. Then write:

$$c_1 \cdot c_2 = (a_1 + ☺b_1)(a_2 + ☺b_2) = a_1a_2 + a_1b_2☺ + a_2b_1☺ + b_1b_2☺^2.$$

If we agree to replace $☺^2$by -1,we have

$$c_1 \cdot c_2 = (a_1a_2 - b_1b_2) + ☺(a_1b_2 + a_2b_1)$$

which is the right answer! We agree to use j instead of ☺, and everywhere j^2appears we replace it with -1. So, in this sense "$j^2 = -1$" but j is just a notational aid.

A.2 Complex-valuled functions

Now that we have introduced complex numbers to our field of operation, we can now extend the notion of a function to include the possibility of a function that takes on values from this complex field. Specifically, we considered a function to be a mapping from an independent variable, say t to the real numbers, such that $f(t) = c$, where $c \in \Re$. By simply extending this notion to include the possibiltiy that the function $f(t) = z$, where $z \in \mathcal{C}$,where $\mathcal{C}$ denotes the field of complex numbers, such that $z = x + jy$, and $x, y \in \Re$

A.3 Complex variables

To this point, we have considered the independent variable in our functions to be taken from either the integers or the reals. However, now that we have extended our field of operation to include the field of complex numbers, it is only natural to consider extending the notion of a function from one that operates not on just real-valued independent variables, but also, possibly, complex-valued variables. A complex variable is simply a variable that can take on values from the field if complex numbers. So the variable $z = x + jy$ is a complex variable and any function of z must be considered carefully, as it is now a function of a complex variable and as such is considerably more complex than a function of a real variable.

A.4 Functions of a complex variable

We can now consider algebraic functions of complex variables by considering the functions as taking algebraic operations on the complex numbers, again treating them as polynomials in j, and then reducing the resulting expression into a single complex number after the algebraic operations are complete. For example, for the function

$$f(z) = z^2 + 1$$

we can simply write

$$f(x + jy) = (x + jy)^2 + (1 + j0) = (x^2 - y^2 + 1) + j(2xy)$$

and see that the resulting value has both a real part and an imaginary part, each of which can be expressed in terms of the real parts and imaginary parts of z. It is often useful to consider the function $f(z)$ in terms of its real part and imaginary part separately, so that each might more simply be written as a real-valued function of two real-valued variables. Specifically, we have

$$f(z) = f(x + jy) = u(x, y) + jv(x, y),$$

where x and y are real-valued variables, and u and v are real-valued functions of two real-valued variables. In this example, we have

$$\begin{aligned} f(z) &= u(x,y) + jv(x,y) = (x^2 - y^2 + 1) + j(2xy), \\ u(x,y) &= x^2 - y^2 + 1, \\ v(x,y) &= 2xy. \end{aligned}$$

Now the function f itself is complex valued, and notions of graphically displaying such functions is no longer as simple as it was for real-valued functions. However we can simply plot the real part, u or the imaginary part v of $f(z)$, to graphically depict its behavior. Similarly, we could consider other representations of the resulting complex number, such as its polar form, and plot the magnitude, $|f|$, and the phase, $\angle f$ over the complex (x, y) plane. As a result, we can then ask whether there is any $(x + jy)$ at which

$$f(x + jy) = (0 + j0) = 0,$$

which is true if and only if $\Re[f] = \Im[f] = 0$, or if or if and only if $|f| = 0$.

We can check this:

$$\begin{aligned} f(x+jy) &= (x+jy)^2 + (1+j0) \\ &= \underbrace{(x^2 - y^2 + 1)}_{\Re\{z\}} + j\underbrace{(2xy)}_{\Im z} \end{aligned}$$

which equals $(0 + j0)$ if and only if $x = 0$ and $y = \pm 1$, i.e. if z is a complex variable, then

$$f(z) = 0 \text{ at } z = (0 \pm 1j) \triangleq \pm j.$$

If $z = x + jy$ is a complex variable, then we can plot $|f(z)|$ as a surface over the $x - y$ plane. It hits zero at the points shown in figureA.2.

We can similarly plot the phase of the function, i.e. we can plot $\angle f(z)$, over the (x, y) plane as shown in figure A.3.

To simplify notation, and to accommodate a broad class of operations using complex numbers, we call the "complex conjugate", denoted z^* , of a complex number z to represent the complex number that has the same real part but an imaginary part with the opposite sign, i.e. for $z = x + jy$, we have $z^* = x - jy$, and $z^* = re^{-j\theta}$. As a result, we can compactly represent the relationship between the magnitude of a complex number and the number itself, i.e. we have $r^2 = z^* z$. We can also obtain $(z + z^*) = 2x$, and $(z - z^*) = j2y$. Complex conjugation distributes over addition and multiplication (and division) of complex numbers, i.e. $(z_1 + z_2)^* = z_1^* + z_2^*$,and $(z_1 z_2)^* = z_1^* z_2^*$.

A.5 Complex Systems

Complex numbers are often used in real systems, even when only real-valued quantities exist in the constituent components of the system itself. Applications such as digital communications (modems), radar, sonar, and computed imaging systems are just a few applications where complex operations are computed using real-valued constituent components.

Consider the linear shift-invariant system described by the flowgraph in figureA.4.

where $x[n] = x_R[n] + jx_I[n]$ and $y[n] = y_R[n] + jy_I[n]$ are complex-valued sequences. We can draw a block diagram that can implement the above system using real signals and real multipliers, adders and delays.

The first step is to realize that $x[n] = x_R[n] + jx_I[n]$ and $y[n] = y_R[n] + jy_I[n]$ are pairs of real-valued sequences, i.e., the system above has two inputs $x_R[n]$ and $x_I[n]$ and two outputs $y_R[n]$and $y_I[n]$. The second step is to recall that complex multiplication and complex addition are defined in terms of real multiplication and real addition, as described previously. We call the output of first multiplication, by $0+j2$, $v[n] = v_R[n] + jv_I[n]$. This multiplication is accomplished as

$$v[n] = j2(x_R[n] + jx_I[n]) = -2x_I[n] + j2x_R[n]$$

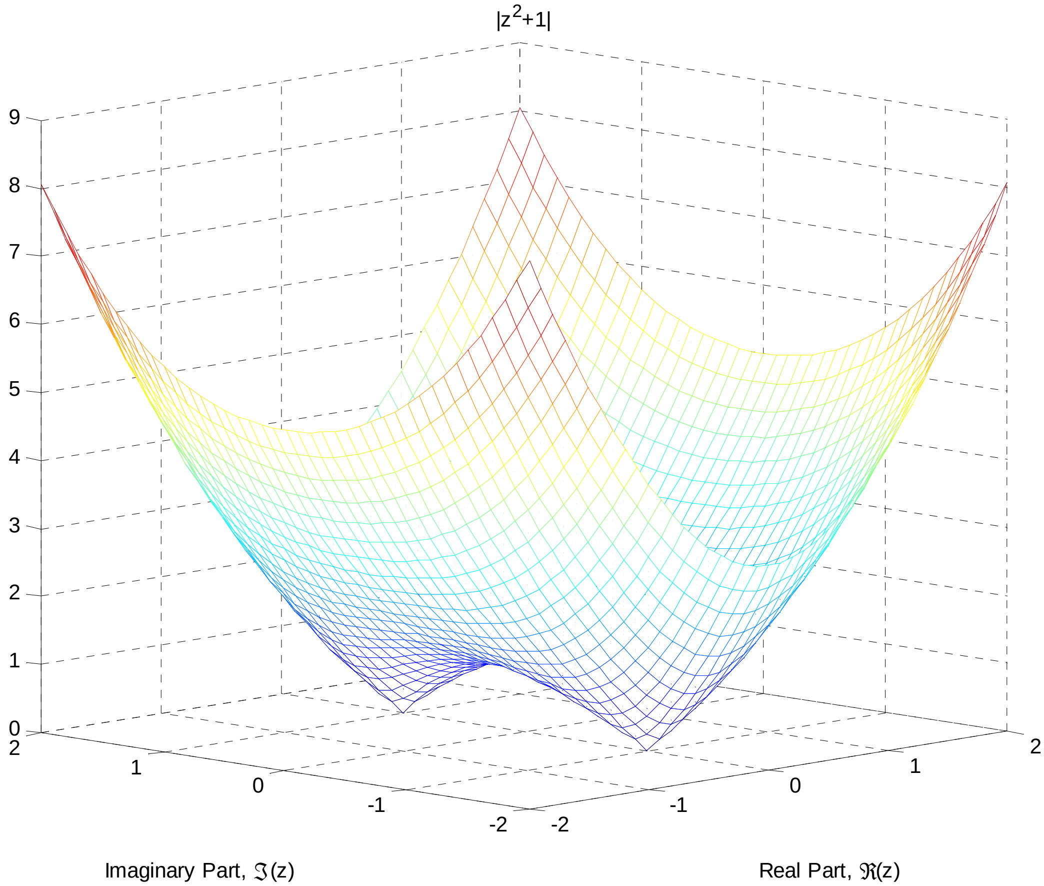


Figure A.2: Magnitude of the function $f(z) = z^2 + 1$,i.e. $|z^2 + 1|$.

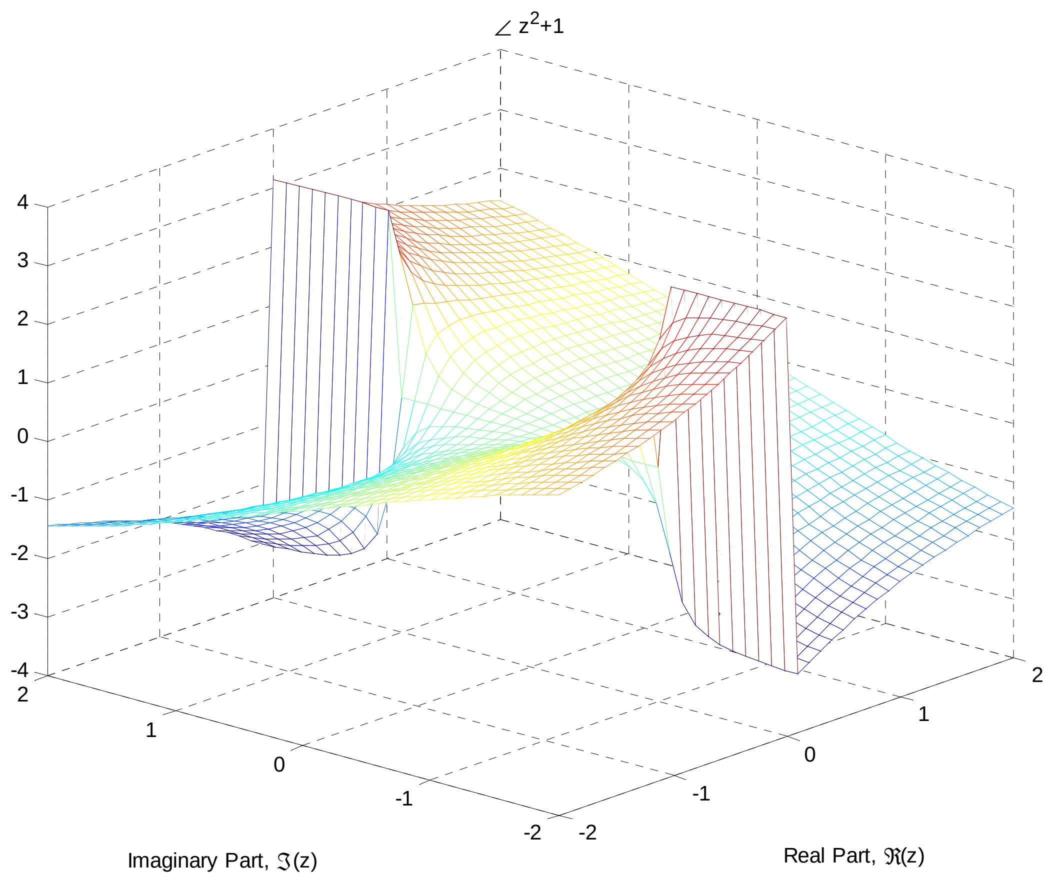


Figure A.3: Phase of the function $f(z) = z^2 + 1$,i.e. $\angle f(z)$.

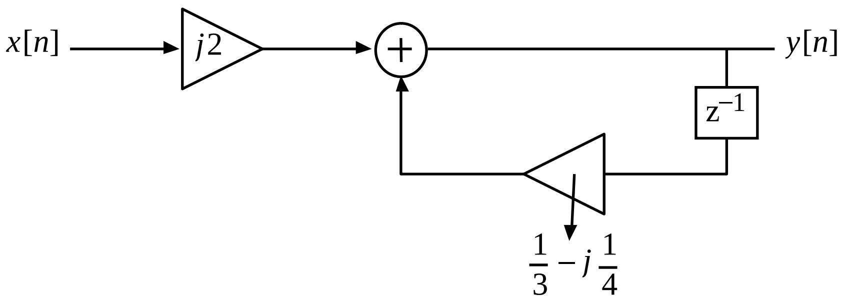


Figure A.4: Representation of a complex-valued linear shift invariant system.

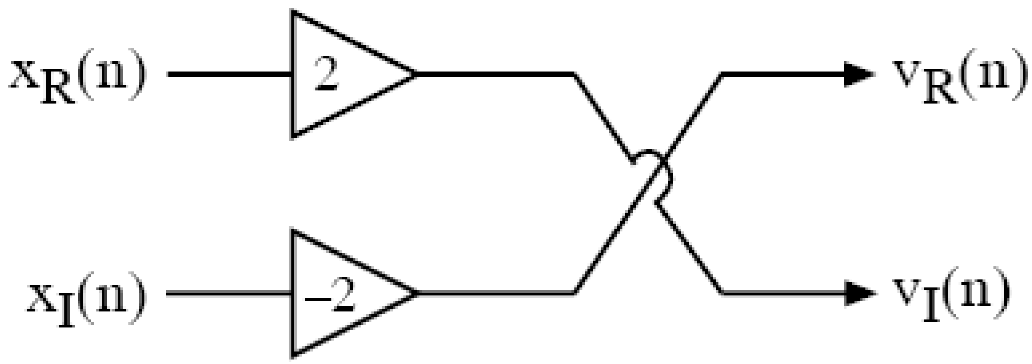
Thus,

$$v_R[n] = -2x_I[n]$$

and

$$v_I[n] = 2x_R[n],$$

which is implemented as



Now, let $w[n]$ be the output of the multiplication by $0 - j$. We see that $w[n]$ can be computed as

$$\begin{aligned} w[n] &= \left(\frac{1}{3} - j\frac{1}{4}\right)(y_R[n-1] + jy_I[n-1]) \\ &= \frac{1}{3}y_R[n-1] + \frac{1}{4}y_I[n-1] + j\left(-\frac{1}{4}y_R[n-1] + \frac{1}{3}y_I[n-1]\right). \end{aligned}$$

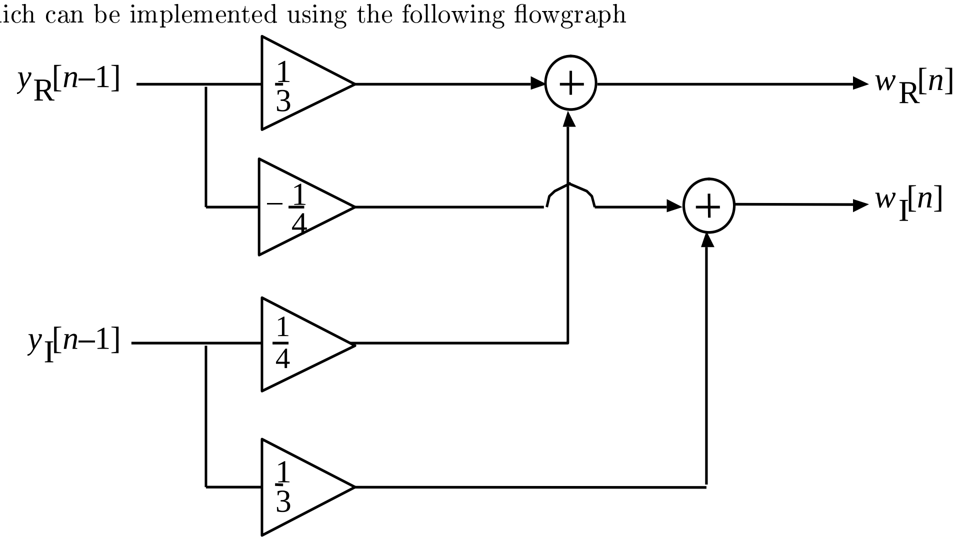
Thus we have,

$$w_R[n] = \frac{1}{3}y_R[n-1] + \frac{1}{4}y_I[n-1]$$

and

$$w_I[n] = -\frac{1}{4}y_R[n-1] + \frac{1}{3}y_I[n-1],$$

which can be implemented using the following flowgraph



Now using that $y[n] = v[n] + w[n]$, so that $y_R[n] = v_R[n] + w_R[n]$ and $y_I[n] = v_I[n] + w_I[n]$, we have the complete block diagram implementation using only real-valued signals as shown in figure A.5below.

The original block diagram is simply a concise way of representing this complicated system.

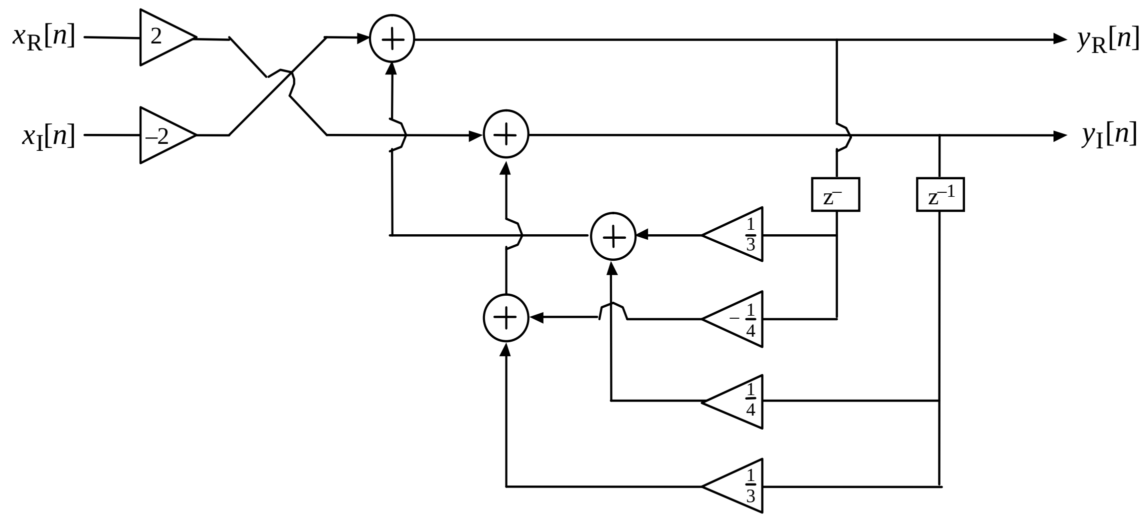


Figure A.5: Flowgraph implementing the system of Figure A.4 using real-valued signals.

A.6 Complex Functions of Complex Variables

Recall that $X(s)$, the Laplace transform as given by $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$, is a complex function of a complex variable. This has a number of important mathematical implications and can be developed in great detail. In this text, however, we will keep our treatment of such functions brief and limited in scope. To aid with the development of the z-transform for sequences, we will briefly review some of the properties of such complex functions. When we say that the function is a "complex function," we simply mean that the function takes on values in the complex numbers. That is has both real and imaginary parts, as well as a magnitude and phase. Similarly, when we say that a function is a function of a complex variable, then the argument of the function takes on values in the complex numbers. For example, consider the complex function of the complex variable

$$f(\underset{\substack{\uparrow \\ \text{complex variable}}}{z}) = z$$

which takes on values exactly equal to its argument. Now, since the function is complex, it has both real and imaginary parts, and since its argument is complex, it also has both real and imaginary parts. As a result, it is difficult to conceptualize, or to plot, the whole function all at once. This is why we often look at one real-valued property of $f(z)$ at a given time, and since the variable is complex, then a single real-valued property can be thought of as a surface over the complex plane. As the variable z takes on all possible values, for which $f(z)$is defined, we can imagine a surface defined by, say, the real-part of $f(z)$. We can now describe the surfaces $\Re f(z), \Im f(z), |f(z)|,$ and$\angle f(z)$ as surfaces over the 2-D complex z-plane.

Recall that

$$f(z) = z = x + jy$$

so we have that the real-part satisifies,

$$\Re f(z) = x$$

which defines a plane through y-axis of the complex z-plane with unit slope. This is shown in figure A.6.

For the imaginary part, we have that

$$\Im f(z) = y$$

which defines a plane through x-axis with unit slope as shown in Figure A.7.

We can also consider the magnitude of the function, $|f(z)|$, which is non-negative and will always lie on or above the complex plane as shown in Figure A.8. Note that for a given value of z, the magnitude is given by

$$|f(z) = \sqrt{x^2 + y^2}.$$

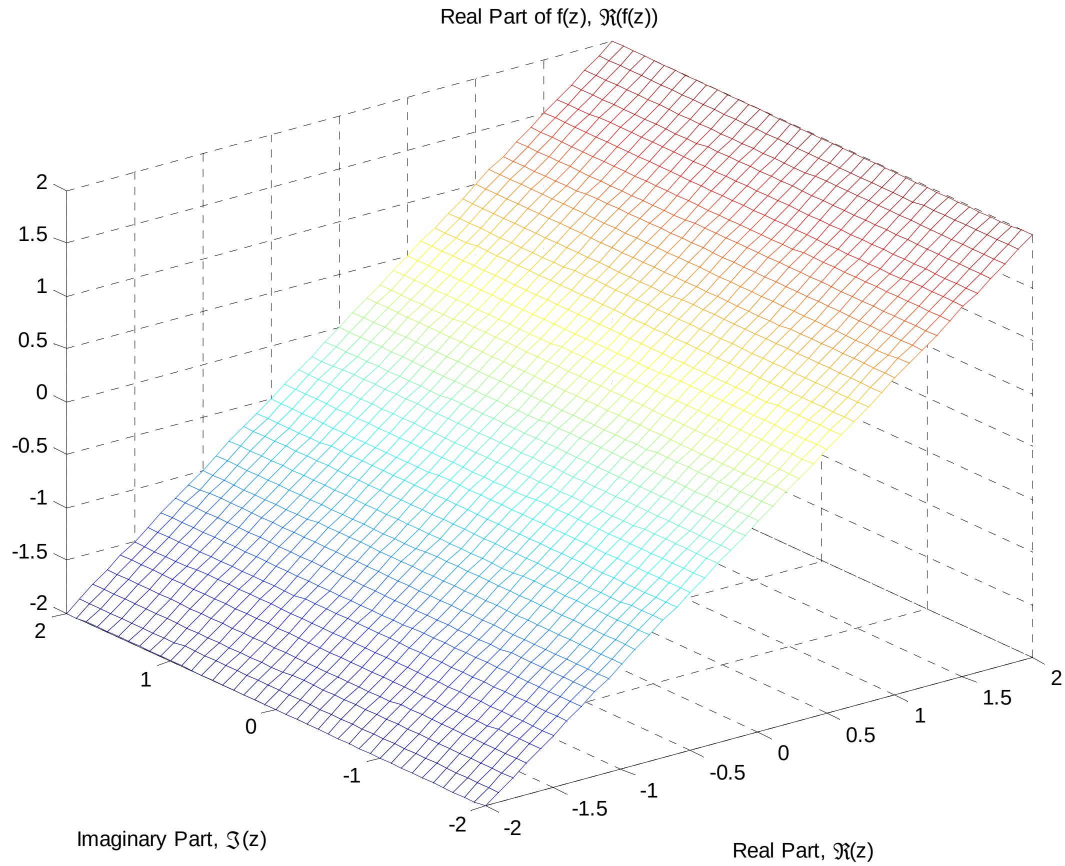


Figure A.6: The surface defined by the real-part of the function $f(z) = z = x + jy$ is shown as a plane over the complex plane, intersecting complex plane along the y-axis, and linearly increasing as a function of x.

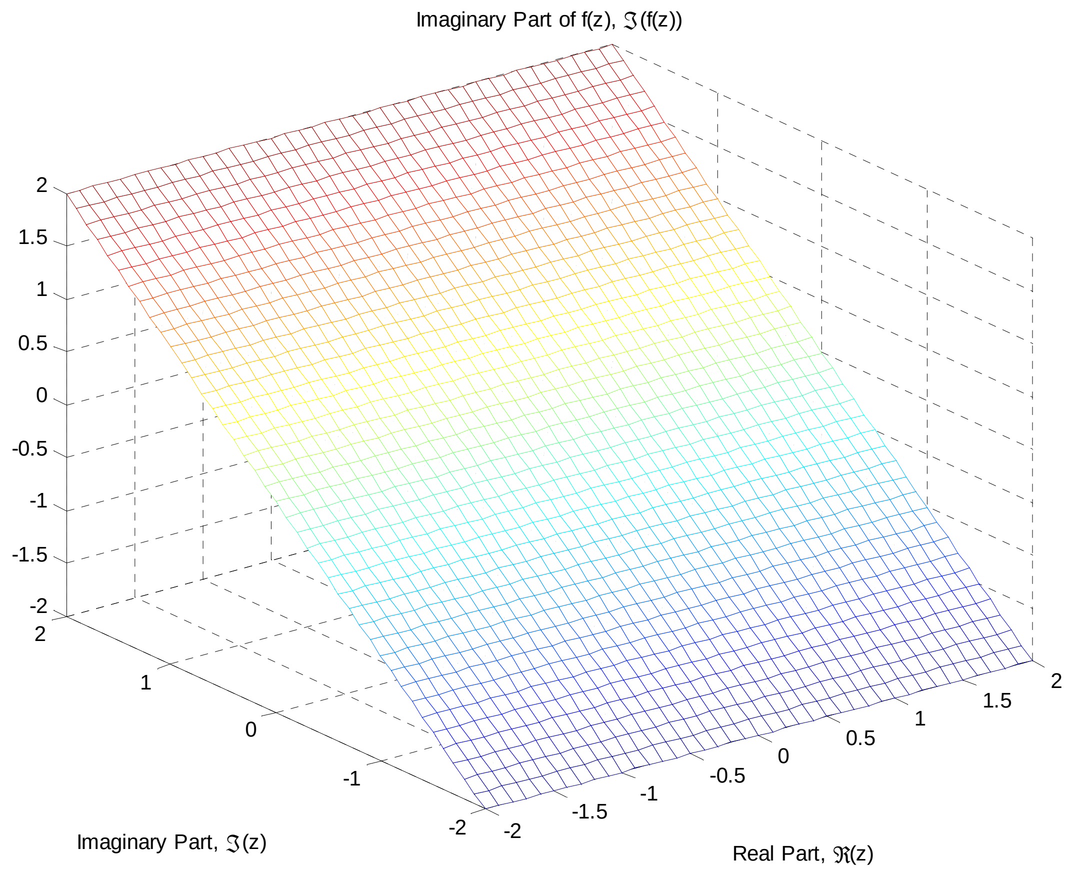


Figure A.7: 2 The surface defined by the imaginary-part of the function $f(z) = z = x + jy$ is shown as a plane over the complex plane, intersecting the complex plane along the x-axis, and linearly increasing as a function of y.

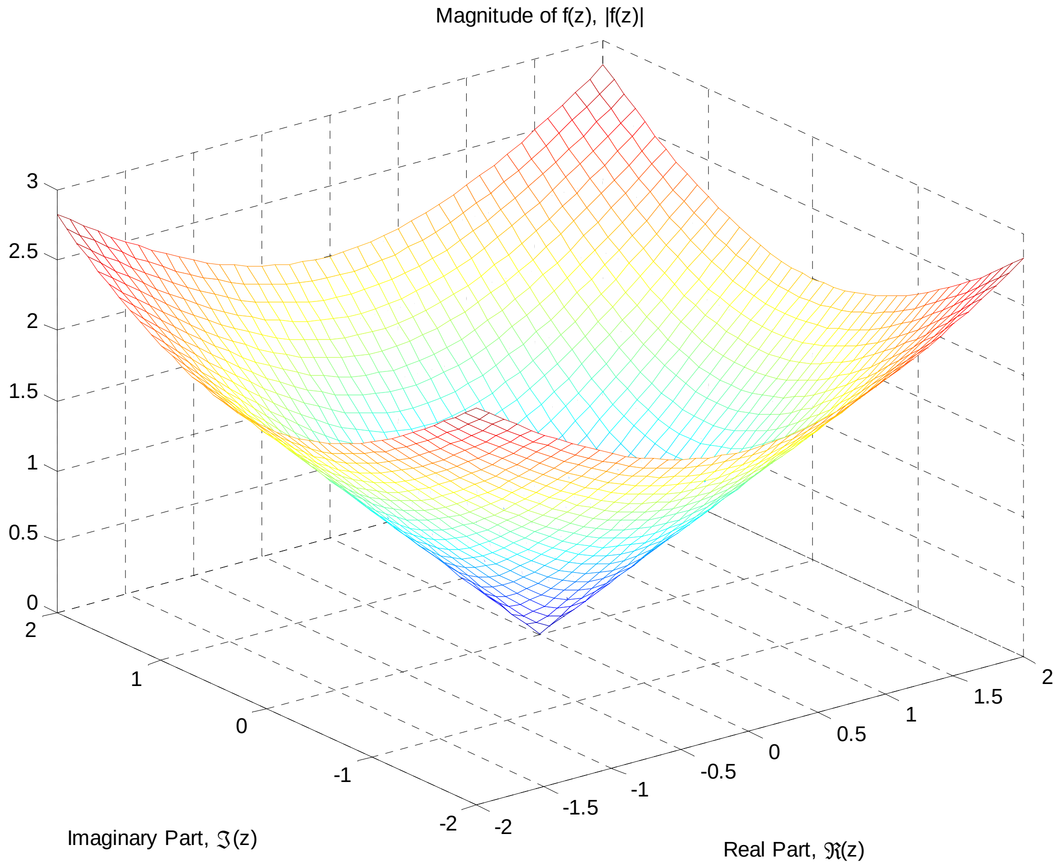


Figure A.8: The surface defined by the magnitude of the function $f(z) = z = x + jy$ is shown as a cone of slope one over the complex plane, centered at the origin.

All values of z that take on the same magnitude would trace out a circle in the complex plane. As the magnitude increases, the radius of the circle also increases. As a result, the surface defined by the magnitude, $|f(z)|$, is an inverted cone, of slope one, centered at the origin.

Finally, the phase of the function is given by

$$\angle f(z) = \arctan\left(\frac{y}{x}\right),$$

which defines a spiral ramp starting at $+x$-axis (which $\angle z$ cuts through), and which ramps up in a counter clockwise direction to the height π along the $-x$-axis. In the clockwise direction from the $+x$-axis, the surface ramps down to the level $-\pi$ along the $-x$-axis. This is shown in Figure A.9.

A real-valued function can be fully described by a plot of the values of the function along an axis describing its independent variable. A complex-valued function of a complex variable can be fully described by the surfaces defined by its real and imaginary parts. Similarly, it can be fully described by the surfaces defined by its magnitude and phase.

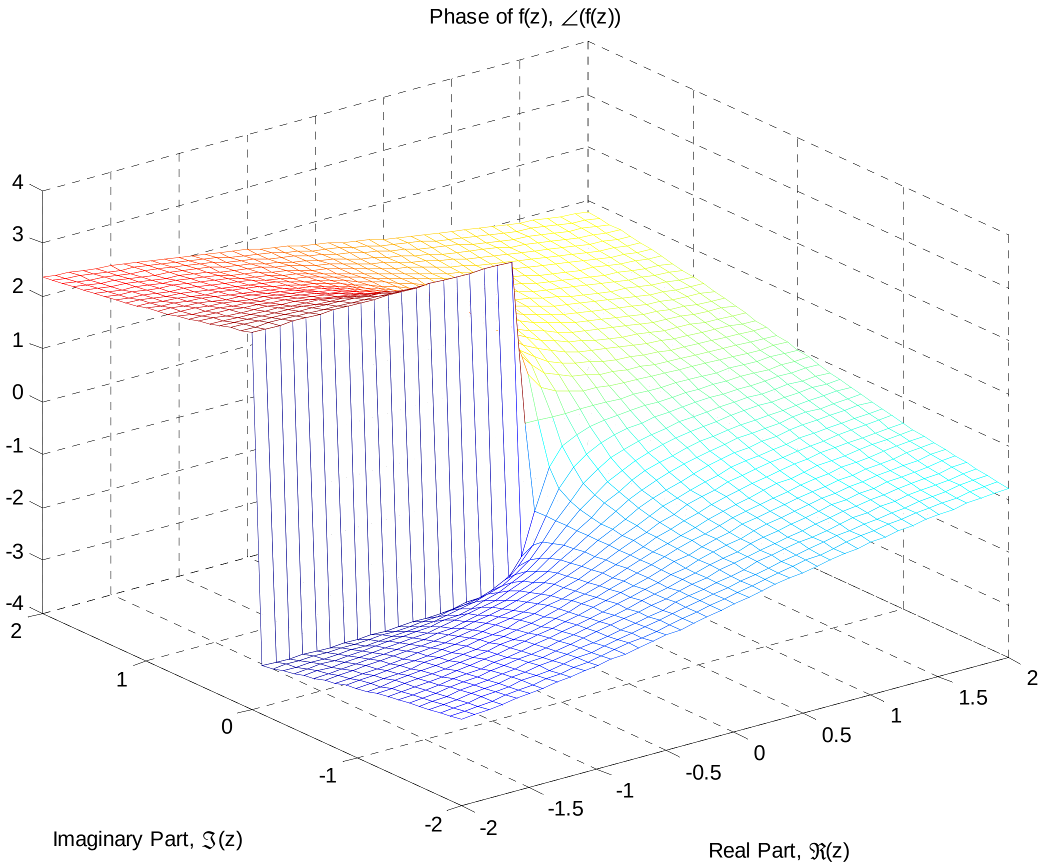


Figure A.9: The surface defined by the phase of the function $f(z) = z = x + jy$ is shown as a spiral ramp through the complex plane.