Appendix D

Appendix: Impulses, samples, and delta's, Oh My!

D.1 The Dirac delta

The Dirac delta is best described as a distribution, rather than as a function, since, strictly speaking, it is not a function. A distribution is defined as follows

> A *distribution* maps a function to a number.

With that out of the way, we can now define the Dirac delta, or impulse as follows

> The *Dirac delta* is the distribution operating on the function $f(t)$, where $f(t)$ is assumed to be continuous near $t = 0$, is given by
>
> $$\delta\{f(t)\} \triangleq f(0).$$

We refer to the Dirac delta interchangeably as an "impulse," and colloquially as a "delta function," even though it is not a function, but rather a distribution. Using this slight abuse of terminology, we also often will replace the form $\delta f(t)$ with the less convenient, but more natural form

$$\int_{-\infty}^{\infty} \delta(t) f(t) dt \triangleq f(0). \tag{D.1}$$

In this form, we can imagine the impulse as having a "sifting property" whereby when placed within an integral, it "sifts out" the value of the integrand at the precise value of $t = 0$. Note that this special form does not provide any additional properties to the impulse other than its definition within an integral. Outside of an integral, an impulse is meaningless, since it is not a function and is only defined by how it operates on a function when placed within an integral of the form above. For the special case of $f(t) = 1$,we have that

$$\int_{-\infty}^{\infty} \delta(t) dt = 1.$$

Unfortunately, now that you are comfortable as accepting an impulse as a distribution, and that it is only defined when placed in an integral of the form D.1, the integral in D.1is not truely an integral. In fact, it is completely defined by D.1. However, we can manipulate D.1and properties of continuous functions $f(t)$ to obtain forms such as

$$\int_{-\infty}^{\infty} \delta(at) f(t) dt = \frac{1}{|a|} \int_{-\infty}^{\infty} \delta(\tau) f(\tau/a) d\tau = \frac{1}{|a|} f(0),$$

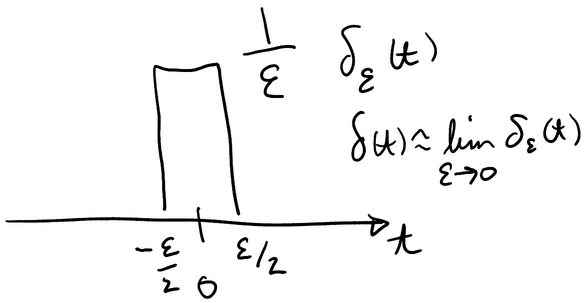


Figure D.1: A conceptual limiting process for the delta function, $\delta(t) \approx \lim_{\epsilon\to 0} \delta_\epsilon(t)$.

through the change of variables $\tau = at$, where the absolute value arises from a combination of replacing $adt = d\tau$. Note that if $a < 0$ then the sign of the integral would change by this substitution, however we would also need to swap the limits of integration also due to this change of variable, leaving the overall sign the same, which is why we can simply use an absolute value and leave the limits unchanged, regardless of the sign. From this we can equate

$$\delta(at) = \frac{1}{|a|}\delta(t).$$

We can also use the shifting property

$$\int_{-\infty}^{\infty} \delta(t-c)f(t)dt = f(c),$$

through a change of variable $\tau = t - c$, which leaves scale and sign of the integral intact. Combining these two approaches yields

$$\int_{-\infty}^{\infty} \delta(at-c)f(t)dt \overset{(\tau=at)}{=} \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(\tau-c)f\big(\tau/a\big)d\tau \overset{(\sigma=\tau-c)}{=} \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(\sigma)f\big((\sigma+c)/a\big)d\sigma = \frac{1}{|a|}f(c/a),$$

which gives the relation

$$\delta(at-c) = \frac{1}{|a|}\delta(t-c/a).$$

The sifting property of the delta function makes one tempted to think of it as a limiting process for a short, tall pulse of the form, $\frac{1}{\epsilon}[u(t+\epsilon/2)-u(t-\epsilon/2)]$, although the limit shown in figure D.1 is not a function, though when the limit of such a function is placed in an integral, the limit of the integral, for certin well-behaved functions $f(t)$, will behave like the impulse distribution.

We can say the following

$$f(0) = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} \delta_\epsilon(t)f(t)dt = \int_{-\infty}^{\infty} \delta(t)f(t)dt,$$

however, saying that $\delta(t) = \lim_{\epsilon\to 0}\delta_\epsilon(t)$ would require bringing the limit inside the integral, which is mathematically incorrect, since the limit $\lim_{\epsilon\to 0}\delta_\epsilon(t)$ does not exist. An impulse is usually pictoraly depicted as an arrow as shown in figure D.2.

where if the impulse is drawn without an "amplitude" it is assumed to have unit area. When multiple impulses arise, as in $\delta(t) + \frac{2}{3}\delta(t-1) + 2\delta(t-2)$, the "amplitude" of the impulse is used to denote the scale factor to be applied to the sifted value of the function at that location, as in figureD.3.

D.2 The Kronecker Delta

In discrete-time systems, we are often interested in forming a sequence that is zero at every sample value, except for a single value, for which the sequence takes on the value one. In mathematics, a function that has

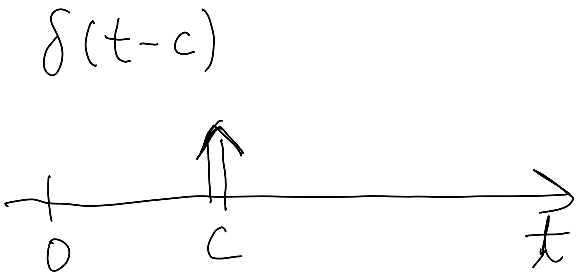


Figure D.2: Graphical depiction of an impulse $\delta(t-c)$.

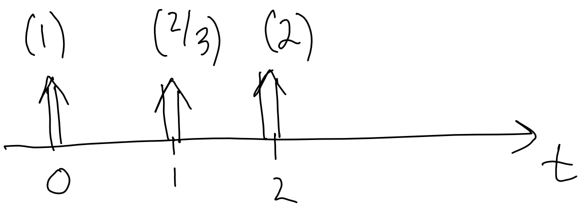


Figure D.3: Graphical depiction of the function $\delta(t) + \frac{2}{3}\delta(t-1) + 2\delta(t-2)$.

a property similar to this one is known as the Kronecker delta function, or Kronecker delta. The Kronecker delta is a function of two variables, i and j and has the property that it is zero for all values of i and j except for the case when $i = j$, i.e., we have

$$\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & \text{otherwise}. \end{cases}$$

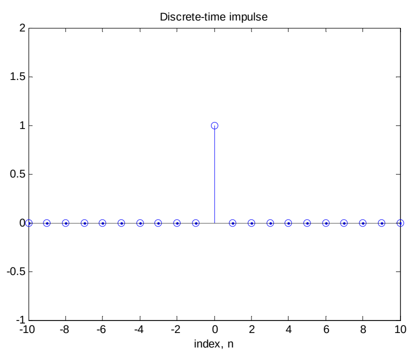
In some texts, a one-argument variant of the Kronecker delta is used that takes the form

$$\delta_k = \begin{cases} 0, & k \neq 0, \\ 1, & k = 0. \end{cases}$$

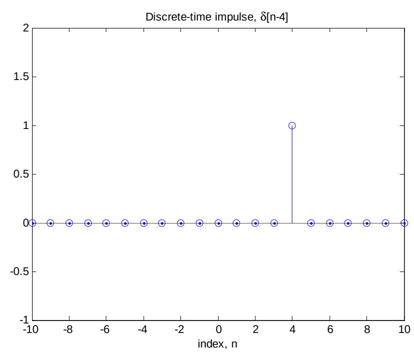
In discrete-time signal processing, we will make extensive use of the Kronecker delta function in a slightly different variation of the one-argument version given above. When doing so, we often refer to this as a sequence, and call it a "unit pulse" sequence, defined as

$$\delta[n] = \begin{cases} 0, & n \neq 0, \\ 1, & n = 0. \end{cases}$$

We note the similarity in notation, and definition of the "unit pulse" with the "impulse" or Dirac delta, $\delta(t)$, and as a result, we often use the term "unit pulse," "discrete-time impulse," and "impulse" interchangeably. A plot of the sequence corresponding to a disrete-time impulse is shown in figure D.4.



(a)



(b)

Figure D.4: A discrete-time impulse, which is also referred to as a unit pulse. In (a), $\delta[n]$ is shown as a single non-zero value, at $n = 0$. In (b), the sequence $\delta[n-4]$ is shown as a single nonzero value at $n = 4$.