ECE 410 **DIGITAL SIGNAL PROCESSING** **D. Munson**
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Classes of Digital Filters

FIR – Finite impulse response;
$\{h_n\}$ finite in length

IIR – Infinite impulse response;
$\{h_n\}$ infinite in length

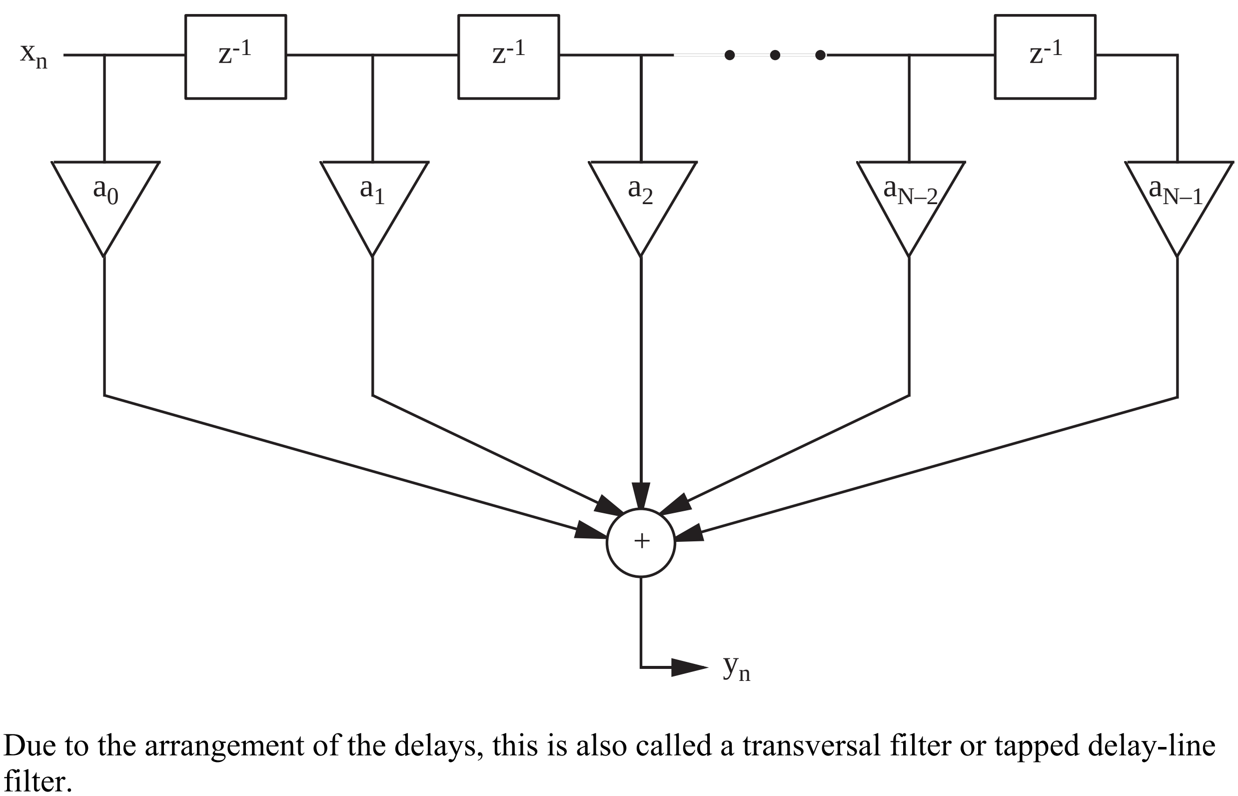
FIR Filter Structures

$$h_n = \{a_0, a_1, a_2, \ldots, a_{N-1}, 0, 0, \ldots\}$$

$$\Rightarrow \quad H(z) = \sum_{n=0}^{N-1} a_n \, z^{-n}$$

TF is a polynomial in z^{-1}

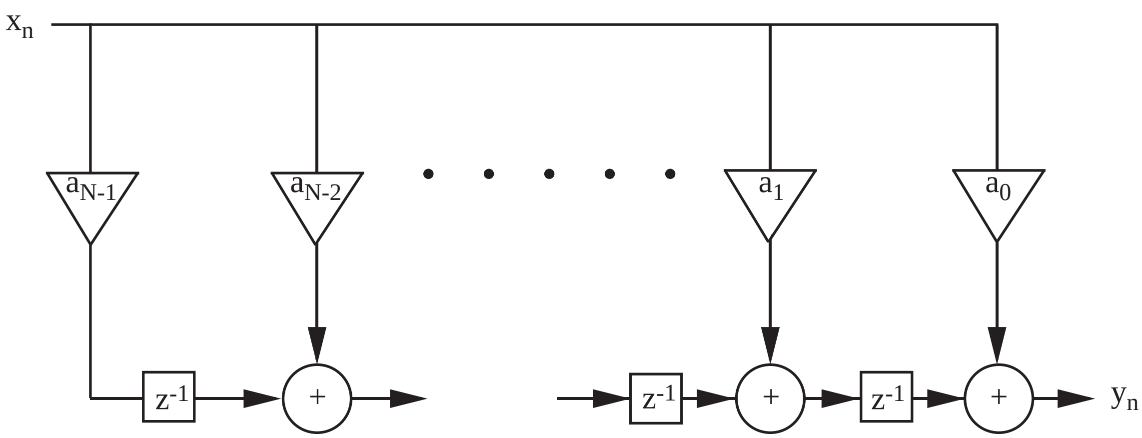
Direct Form Structure:



Due to the arrangement of the delays, this is also called a transversal filter or tapped delay-line filter.

The implementation of a transfer function is not unique. The transfer function describes only the input-output properties of the system. For any transfer function, there are an infinite number of possible realizations of that transfer function. For example, consider the transpose-form structure.

Transpose Form:
(obtained by reversing all flows)



This structure has the same transfer function as the Direct Form structure and is very commonly used.

Advantage: Easier to fully parallelize. No adder tree at output as in Direct Form.

FIR filters are nearly always implemented <u>nonrecursively</u> as in the above diagrams. Theoretically, though, FIR filters can be recursive, as shown in the following example.

Example

Disguise the FIR transfer function as a rational function with non-unity denominator:

$$H(z) = a_0 + a_1 z^{-1} + a_2 z^{-2}$$

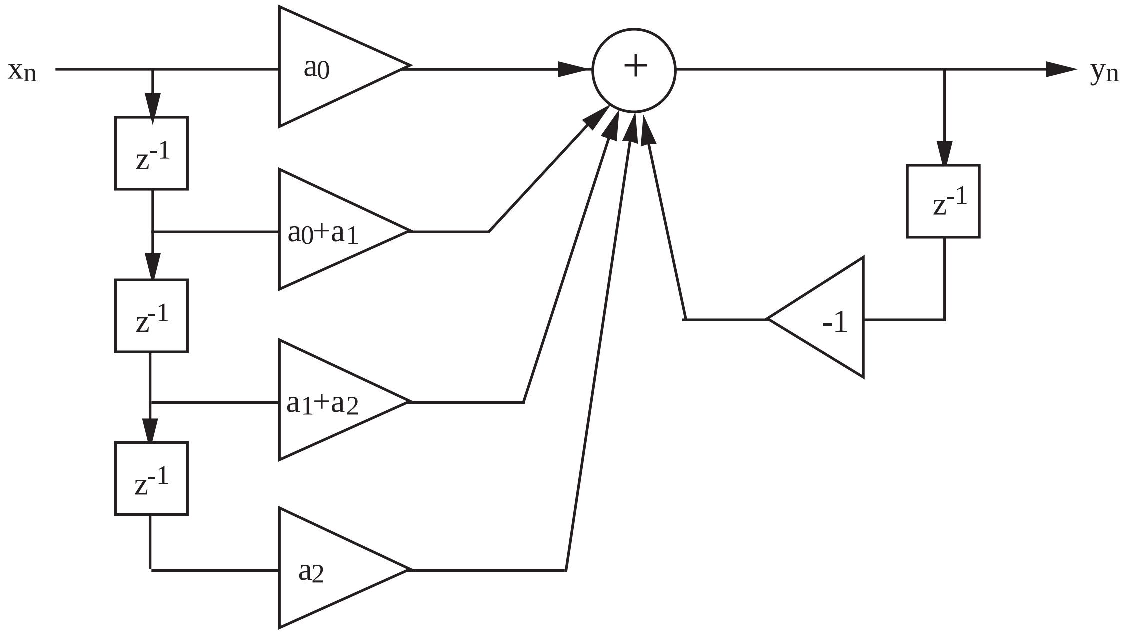
$$= \underbrace{\frac{1+z^{-1}}{1+z^{-1}}}_{\text{for example}} \left(a_0 + a_1 z^{-1} + a_2 z^{-2}\right)$$

$$= \frac{a_0 + z^{-1}(a_0 + a_1) + z^{-2}(a_1 + a_2) + a_2 z^{-3}}{1+z^{-1}}$$

$$\Rightarrow Y(z)\left[1+z^{-1}\right] = [\text{NUM}]\, X(z)$$

$$\Rightarrow Y(z) = -z^{-1}\, Y(z) + [\text{NUM}]\, X(z)$$

Filter structure:



This is a recursive structure (y_n depends directly on y_{n-1}) that realizes the transfer function H(z).

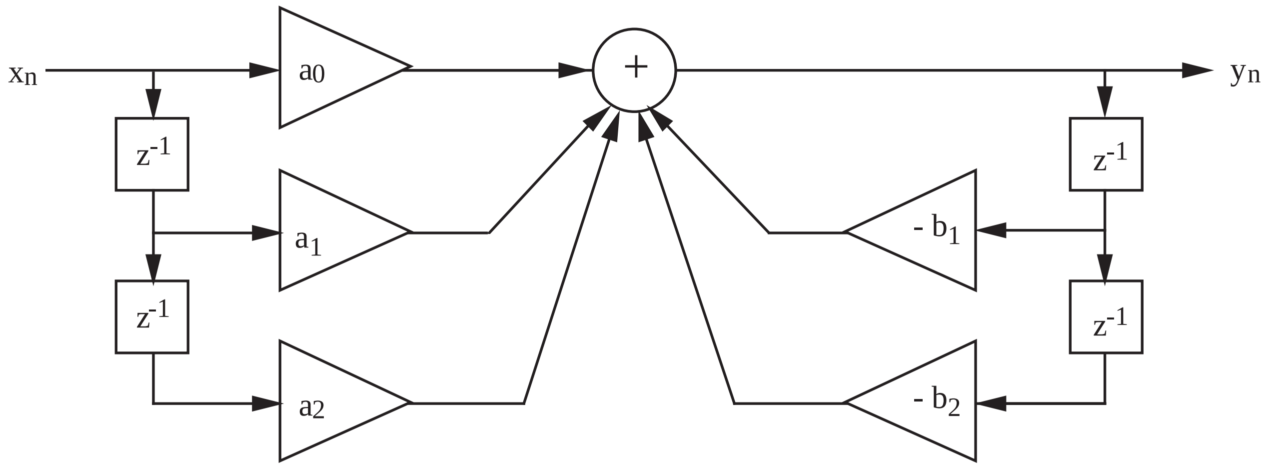
IIR Filter Structures

Transfer functions of IIR filters are not polynomials. We consider rational TF's. IIR filters must be recursive (otherwise they would require an infinite number of adders, multipliers, and delays).

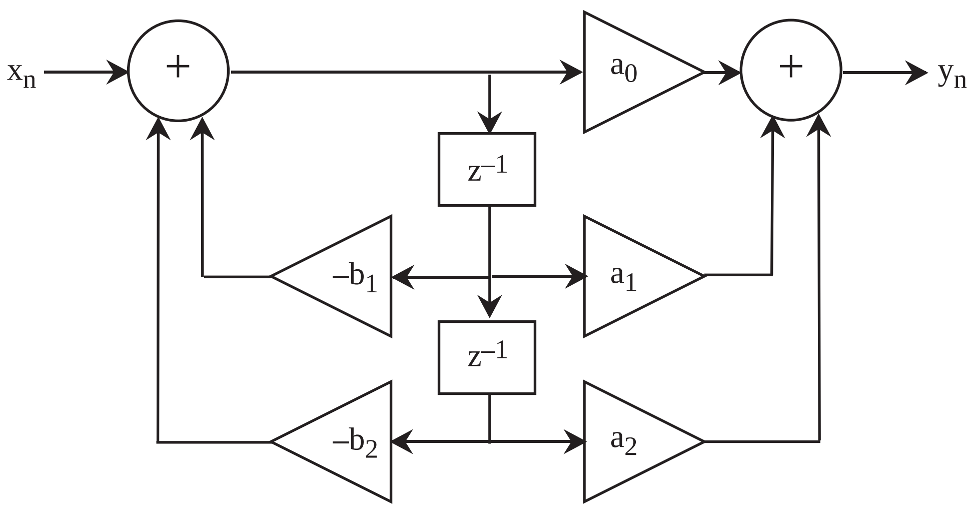
Consider a 2nd-order case:

$$H(z)= \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

Direct Form 1 structure:



Can also implement using a Direct Form 2 structure:



We showed earlier that this structure has the same transfer function H(z) as the Direct Form 1 structure.

IIR filters are always recursive. FIR filters are implemented in nonrecursive form.

Implementation of Higher-Order Digital Filters

(Order of filter = max {degree (Num), degree (Den)} = # delays required for a Direct Form 2 implementation)

High-order direct-form filters can have large error at the output due to multiplication roundoff. Also, the actual $H_d(\omega)$ may deviate considerably from the desired due to coefficient rounding.

Cascaded or parallel second-order sections exhibit smaller error than direct form. Also, splitting into lower-order sections can make filter easier to parallelize (e.g., second-order filter on a chip).

Cascade Form:

$$H(z) = \frac{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}{1 + b_1 z^{-1} + \ldots + b_N z^{-N}}$$

$$= \frac{a_0 z^N + a_1 z^{N-1} + \ldots + a_N}{z^N + b_1 z^{N-1} + \ldots + b_N}$$

Write as:

$$H(z) = a_0 \prod_{i=1}^{N} \frac{z - z_i}{z - p_i}$$

Since a_i, b_i are real, if p_i is complex, there must be some $p_k = p_i^*$.

Pair up poles and zeros so that (assume N is even)

$$H(z) = a_0 \prod_{i=1}^{N/2} H_i(z)$$

where

$$H_i(z) = \frac{(z - z_k)(z - z_\ell)}{(z - p_m)(z - p_n)}$$

is a second-order filter section with $z_\ell = z_k^*$ and $p_n = p_m^*$.

Pair up complex conjugates so that all multiplier coefficients of the second-order sections are <u>real</u>, so filter can use real arithmetic. (If z_k is real, then pair up with any real z_ℓ; similarly for poles.)

For instance, suppose you factor H(z) and find two poles are

$$p_m = 1 + j, \;\; p_n = 1 - j$$

Then pair these poles together in the same $H_i(z)$ so that:

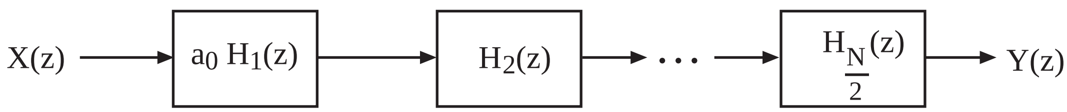
$$H_i(z) = \frac{(z - z_k)(z - z_\ell)}{[z - (1 + j)][z - (1 - j)]}$$

$$= \frac{(z - z_k)(z - z_\ell)}{z^2 - z - jz - z + jz + (1 + j)(1 - j)}$$

$$= \frac{(z-z_k)(z-z_\ell)}{z^2 - 2z + 2}$$

real filter coeffs.

H(z) implemented in cascade form looks like:



where each $H_i(z)$ is a second-order section. If N is odd, then one of the above sections will be a first-order filter.

Parallel Form:

Expand H(z) in a PFE:

$$\frac{H(z)}{z} = \frac{A}{z} + \frac{B_1}{z-p_1} + \ldots + \frac{B_N}{z-p_N}$$

$$\Rightarrow \qquad H(z) = A + \frac{B_1 z}{z-p_1} + \ldots + \frac{B_N z}{z-p_N}$$

Again, pair up complex poles.

If $p_k = p_\ell^*$ then know that $B_k = B_\ell^*$ so that

$$\frac{B_k z}{z-p_k} + \frac{B_\ell z}{z-p_\ell} = \frac{B_\ell^* z}{z-p_\ell^*} + \frac{B_\ell z}{z-p_\ell}$$

$$= \frac{B_\ell^* z(z-p_\ell) + B_\ell z(z-p_\ell^*)}{\left(z-p_\ell^*\right)(z-p_\ell)}$$

$$= \frac{z^2\left(B_\ell^* + B_\ell\right) - z\left(B_\ell^* p_\ell + B_\ell p_\ell^*\right)}{z^2 - z\left(p_\ell^* + p_\ell\right) + p_\ell p_\ell^*}$$

coeffs. are all real

Parallel realization (assume N is even):

$$H(z) = A + \sum_{i=1}^{N/2} H_i(z)$$

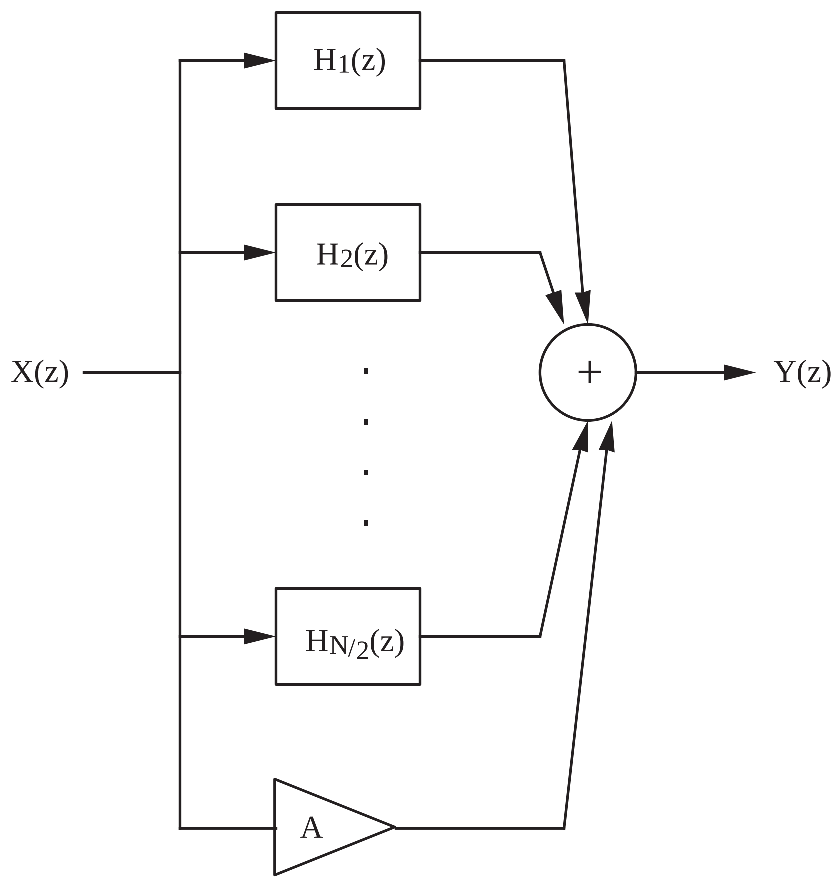
where

$$H_i(z) = \frac{a_{1i} z^2 + a_{2i} z}{z^2 + b_{1i} z + b_{2i}}$$

$$= \frac{a_{1i} + a_{2i} z^{-1}}{1 + b_{1i} z^{-1} + b_{2i} z^{-2}}$$

are second-order sections. Note that, due to the form of the numerator, each of these second-order sections requires one fewer multiplication than for cascade form.

Implementation:



If N is odd, then one of the above filter sections will be first-order.

Example

Suppose $H(z) = \dfrac{z^4 + 1}{z^4 - \dfrac{1}{16}}$.

a) Draw a cascade structure of two second-order sections implementing H(z).

b) Repeat a), but for parallel form.

Solution

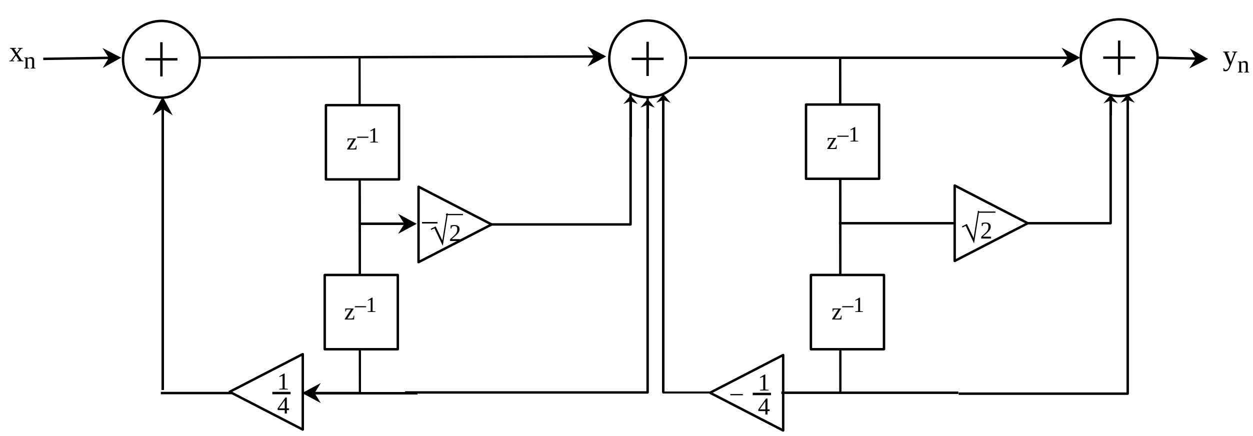
$$\text{a)}\ H(z) = \frac{\left(z - e^{j\frac{\pi}{4}}\right)\left(z - e^{-j\frac{\pi}{4}}\right)\left(z + e^{j\frac{\pi}{4}}\right)\left(z + e^{-j\frac{\pi}{4}}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{2}\right)\left(z - \frac{j}{2}\right)\left(z + \frac{j}{2}\right)}$$

$$= \frac{z^2 - \left(2\cos\frac{\pi}{4}\right)z + 1}{z^2 - \frac{1}{4}} \quad \frac{z^2 + \left(2\cos\frac{\pi}{4}\right)z + 1}{z^2 + \frac{1}{4}}$$

$$= \frac{z^2 - \sqrt{2}\,z + 1}{z^2 - \frac{1}{4}} \quad \frac{z^2 + \sqrt{2}\,z + 1}{z^2 + \frac{1}{4}}$$

$$= \frac{1 - \sqrt{2}\,z^{-1} + z^{-2}}{1 - \frac{1}{4}\,z^{-2}} \quad \frac{1 + \sqrt{2}\,z^{-1} + z^{-2}}{1 + \frac{1}{4}\,z^{-2}}$$

Using direct-form-2 second-order sections, the cascade structure is



b) $$\frac{H(z)}{z} = \frac{z^4+1}{z\left(z-\frac{1}{2}\right)\left(z+\frac{1}{2}\right)\left(z-\frac{j}{2}\right)\left(z+\frac{j}{2}\right)}$$

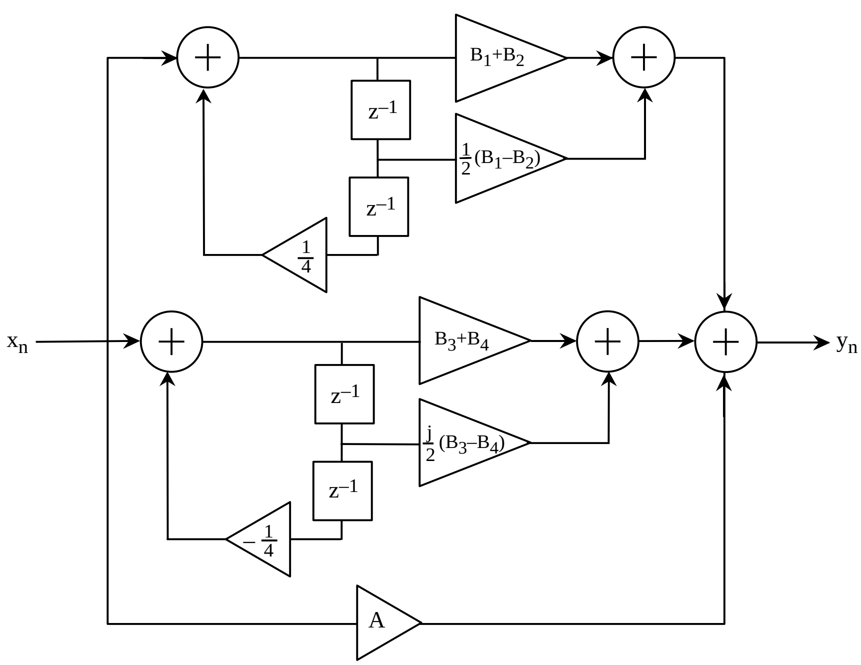
$$= \frac{A}{z} + \frac{B_1}{z-\frac{1}{2}} + \frac{B_2}{z+\frac{1}{2}} + \frac{B_3}{z-\frac{j}{2}} + \frac{B_4}{z+\frac{j}{2}}$$

$$\Rightarrow H(z) = A + \frac{B_1\, z}{z-\frac{1}{2}} + \frac{B_2\, z}{z+\frac{1}{2}} + \frac{B_3\, z}{z-\frac{j}{2}} + \frac{B_4\, z}{z+\frac{j}{2}}$$

$$= A + \frac{(B_1+B_2)\,z^2 + \frac{1}{2}(B_1-B_2)\,z}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{2}\right)} + \frac{(B_3+B_4)\,z^2 + \frac{j}{2}(B_3-B_4)\,z}{\left(z-\frac{j}{2}\right)\left(z+\frac{j}{2}\right)}$$

$$= A + \frac{(B_1+B_2)+\frac{1}{2}(B_1-B_2)\,z^{-1}}{1-\frac{1}{4}z^{-2}} + \frac{(B_3+B_4)+\frac{j}{2}(B_3-B_4)\,z^{-1}}{1+\frac{1}{4}z^{-2}}$$

Using direct-form-2 second-order sections, the parallel structure is



Note: In this diagram, A and the B_i are the coefficients in the PFE. B_4 is the complex conjugate of B_3, so all multiplier values are real. Both the cascade and parallel structures will implement the original transfer function H(z), and will generally do so with less error due to finite register length than a 4th-order direct-form implementation.

Generalized Linear Phase Filters

Linear Versus Generalized Linear Phase

Will say $H_d(\omega)$ is linear phase if

$$H_d(\omega) = \underbrace{|H_d(\omega)|}_{\text{nonnegative}}\, e^{-j\omega M}$$

Fact:

A digital filter doesn't usually have exactly linear phase. But is easy to design FIR filters having what we will call generalized linear phase. Are two types.

Type 1:

$$H_d(\omega) = \underset{\substack{\uparrow \\ \text{real, but not nonnegative}}}{R(\omega)}\, e^{-j\omega M}$$

Type 2:

$$H_d(\omega) = R(\omega)\, e^{j(\alpha-\omega M)}$$

with $\alpha \neq 0$.

We will see that generalized linear phase corresponds to having linear phase over the passband.

FIR Versus IIR Filters

Advantage of FIR: Easy to design with generalized linear phase (linear phase over passband).

Advantage of IIR: Can't have exactly linear phase or generalized linear phase, but IIR can often meet $|H_d(\omega)|$ specification with a much lower order filter.

Generalized Linear Phase Property of FIR Filters

Type 1 Generalized Linear Phase

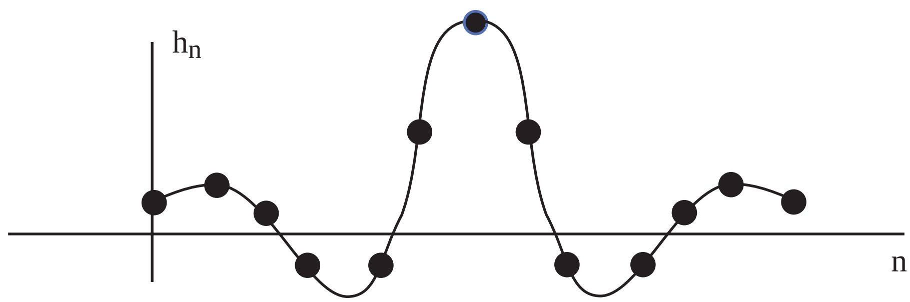
Theorem:

An FIR filter with real-valued unit pulse response $\{h_n\}_{n=0}^{N-1}$ has Type 1 generalized linear phase with $H_d(\omega) = R(\omega)\, e^{-j\omega M}$ iff

$$h_n = h_{N-1-n} \quad \begin{cases} n = 0, 1, \ldots, \dfrac{N}{2} - 1 & \text{(N even)} \\ n = 0, 1, \ldots, \dfrac{N-1}{2} & \text{(N odd)} \end{cases}$$

where $M = \dfrac{N-1}{2}$ and $R(\omega)$ is real and even.

Picture:



Proof:

We give the proof in one direction only. Assuming filter coefficients with even symmetry, we show that $H_d(\omega)$ has the stated form. Now, assume N is odd. Let $M = \dfrac{N-1}{2}$. Given $h_n = h_{N-1-n}$, show that $H_d(\omega)$ has Type 1 generalized linear phase.

$$H_d(\omega) = \sum_{n=0}^{N-1} h_n\, e^{-j\omega n}$$

$$= \sum_{n=0}^{M-1} h_n\, e^{-j\omega n} + h_M\, e^{-j\omega M} + \sum_{n=M+1}^{N-1} h_n\, e^{-j\omega n}$$

$$= e^{-j\omega M} \left[h_M + \sum_{n=0}^{M-1} h_n\, e^{-j\omega(n-M)} + \sum_{n=M+1}^{N-1} h_n\, e^{-j\omega(n-M)} \right] \qquad (\square)$$

Now, making the change of variable n = N–1–k and using $M = \frac{N-1}{2}$, the second sum in (❑) can be written as

$$\sum_{k=M-1}^{0} h_{N-1-k}\, e^{-j\omega(M-k)}$$

Thus,

$$H_d(\omega) = e^{-j\omega M}\left[h_M + \sum_{n=0}^{M-1}\left(h_n\, e^{-j\omega(n-M)} + h_{N-1-n}\, e^{-j\omega(M-n)}\right)\right] \qquad (❑❑)$$

and using $h_{N-1-n} = h_n$, we have

$$H_d(\omega) = e^{-j\omega M}\underbrace{\left[h_M + 2\sum_{n=0}^{M-1} h_n \cos\omega(n-M)\right]}_{\triangleq\, R(\omega)\ \sim\ \text{real valued}}$$

Note:

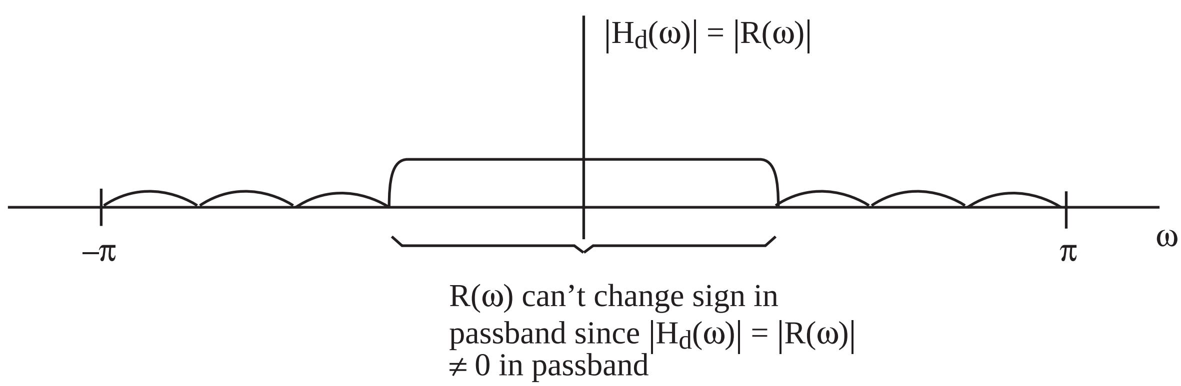
$$|H_d(\omega)| = |R(\omega)|$$

$$\angle H_d(\omega) = \begin{cases} -\omega M & \{\omega : R(\omega) > 0\} \\ -\omega M \pm \pi & \{\omega : R(\omega) < 0\} \end{cases} \qquad (\Delta)$$

$$\uparrow$$
$$-1 = e^{\pm j\pi}$$

(Δ) ⇒ phase is linear except where R(ω) changes sign, in which case the phase jumps by π.

This implies that a generalized linear-phase filter has linear phase over the passband since



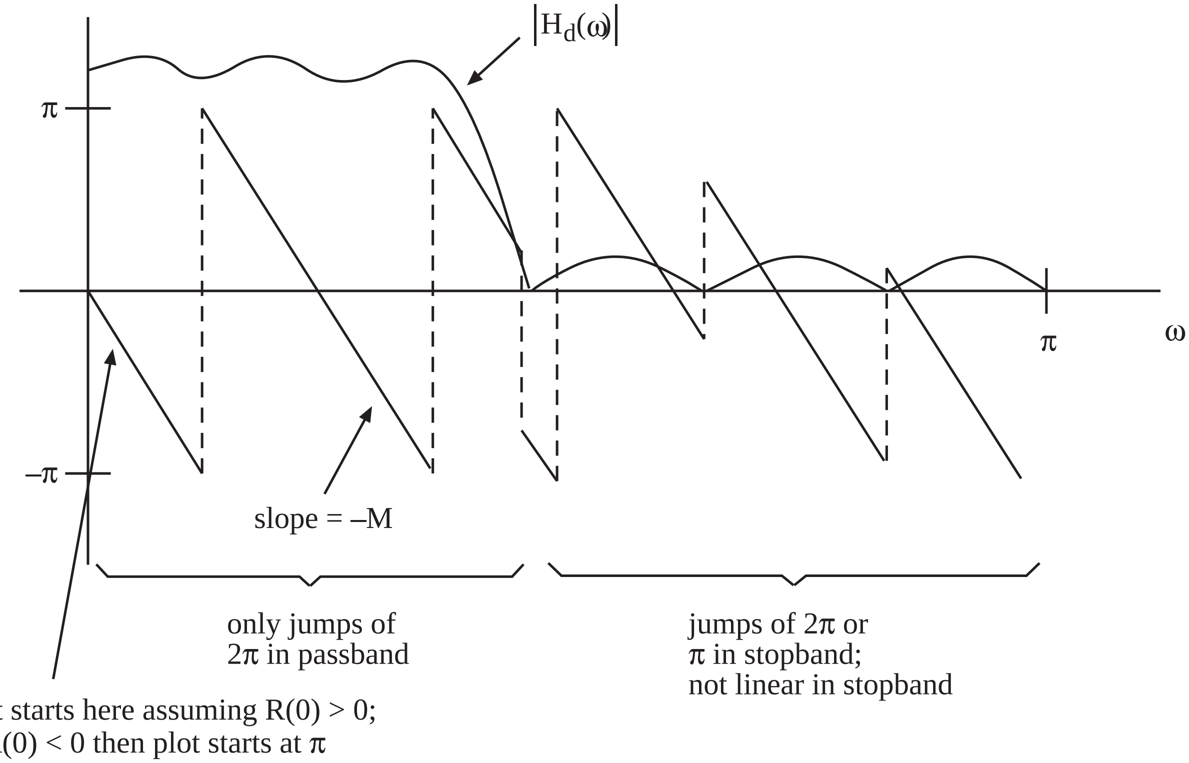
To prove the theorem in the other direction, start with

$$H_d(\omega) = \underset{\substack{\uparrow \\ \text{real and even}}}{R(\omega)}\, e^{-j\omega M} .$$

Then can show $h_n = h_{N-1-n}$.

Note: If N is even then we still take $M = \frac{N-1}{2}$ and the proof is nearly the same as above.

Phase characteristic of a generalized linear phase FIR filter:



In the next lecture, we will consider Type 2 generalized linear phase:

$$H_d(\omega) = \underset{\substack{\uparrow \\ \text{real, odd}}}{R(\omega)}\, e^{j(\alpha-\omega M)}$$

with $\alpha \neq 0$.

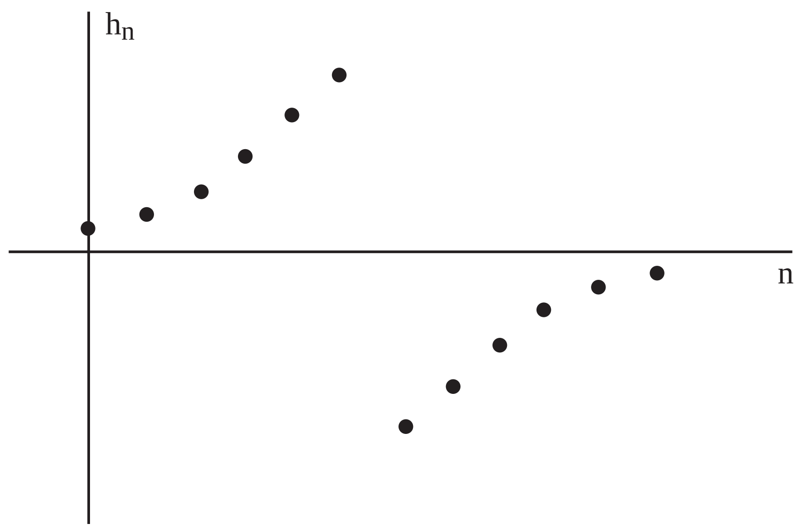
In this case, for a filter $\{h_n\}_{n=0}^{N-1}$ and $\alpha \neq 0$, can show must have $\alpha = \frac{\pi}{2}$ and have odd coefficient symmetry, i.e.,

$$h(n) = -h(N-1-n)$$

and
$$\angle H_d(\omega) = \begin{cases} \frac{\pi}{2} - \omega M & \{\omega : R(\omega) > 0\} \\ \frac{-\pi}{2} - \omega M & \{\omega : R(\omega) < 0\} \end{cases}$$

with $M = \frac{N-1}{2}$ for N even and N odd.

Picture:



Type 2 Generalized Linear Phase

This type of generalized linear phase corresponds to antisymmetric (odd), rather than symmetric (even) filter coefficients.

Theorem:

An FIR filter with real-valued unit pulse response $\{h_n\}_{n=0}^{N-1}$ has Type 2 generalized linear phase with $H_d(\omega) = R(\omega)\, e^{j\left(\frac{\pi}{2}-\omega M\right)}$ iff

$$h_n = -\,h_{N-1-n}$$

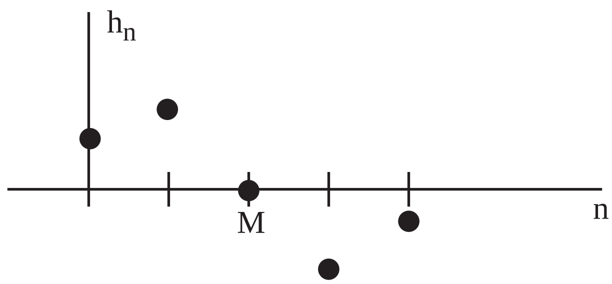
where $M = \frac{N-1}{2}$ and $R(\omega)$ is real and <u>odd</u>.

<u>Proof</u>:

We give the proof in one direction only. Assuming filter coefficients with odd symmetry, we show that $H_d(\omega)$ has the stated form. Now, assuming N is odd and taking $M = \frac{N-1}{2}$ we have from before:

$$H_d(\omega) = e^{-j\omega M}\left[h_M + \sum_{n=0}^{M-1}\left(h_n\, e^{-j\omega(n-M)} + h_{N-1-n}\, e^{-j\omega(M-n)}\right)\right] \qquad (❑❑)$$

Given that $h_n = -h_{N-1-n}$ we must have $h_M = 0$. Can see this pictorially:



Obviously, we cannot have odd coefficient symmetry unless $h_M = 0$.

Now, setting $h_M = 0$ and using $h_{N-1-n} = -h_n$ in (❑❑), we have

$$H_d(\omega) = e^{-j\omega M}\left[\sum_{n=0}^{M-1} h_n\left(e^{-j\omega(n-M)} - e^{j\omega(n-M)}\right)\right]$$

$$= e^{-j\omega M}(-j2)\sum_{n=0}^{M-1} h_n \sin\omega(n-M)$$

$$= e^{j\left(\frac{\pi}{2}-\omega M\right)}\underbrace{\left(-2\sum_{n=0}^{M-1} h_n \sin\omega(n-M)\right)}$$

= $R(\omega)$, which is real and <u>odd</u>

So, for the antisymmetric coefficient case, we have $|H_d(\omega)| = |R(\omega)|$, but now $R(\omega)$ is a linear combination of sines (odd) instead of cosines (even) and

$$\angle H_d(\omega) = \begin{cases} \frac{\pi}{2} - \omega M & \{\omega : R(\omega) > 0\} \\ \frac{-\pi}{2} - \omega M & \{\omega : R(\omega) < 0\} \end{cases}$$

To prove the theorem in the other direction, start with

$$H_d(\omega) = \underset{\substack{\uparrow \\ \text{real and odd}}}{R(\omega)}\, e^{j\left(\frac{\pi}{2}-\omega M\right)}$$

Then can show $h_n = -\,h_{N-1-n}$.

<u>Note</u>: If N is even instead of odd, we still take $M = \frac{N-1}{2}$ and the proof is nearly the same as above.

Example

Determine whether a filter with the unit-pulse response

$$\{h_n\} = \{\underset{\uparrow}{1}, -1, 1\}$$

has generalized linear phase, and if so, whether it has linear phase.

Note: Linear phase $\Rightarrow$ generalized linear phase, but not vice versa.

<u>Solution</u>

h_n is symmetric about its midpoint $\Rightarrow$ Have Type 1 generalized linear phase. To check whether we have linear phase, we must find the phase of the frequency response:

$$H_d(\omega) = 1 - e^{-j\omega} + e^{-j2\omega}$$

$$= \underset{\substack{\uparrow\ e^{-j\omega M} \text{ where } M = 1 \text{ in this example}}}{e^{-j\omega}} \left(e^{j\omega} - 1 + e^{-j\omega}\right)$$

$$= e^{-j\omega} \underbrace{(2\cos\omega - 1)}_{R(\omega)}$$

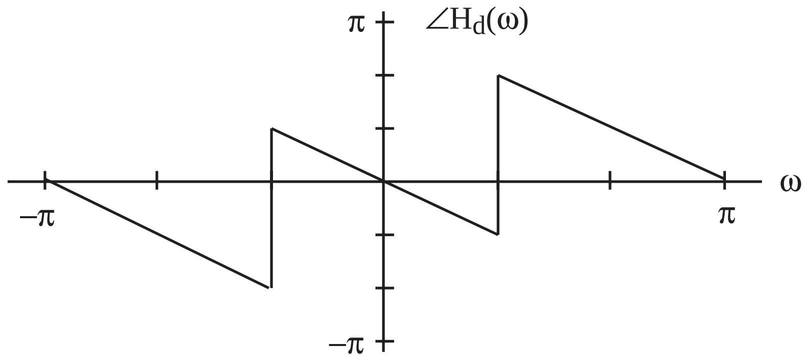
We see that $R(\omega)$ changes sign on $-\pi < \omega < \pi$. Thus, we know that this filter does <u>not</u> have linear phase. Let's find the phase:

$$\angle H_d(\omega) = \begin{cases} -\omega & \{\omega : 2\cos\omega - 1 > 0\} \\ -\omega + \pi & \{\omega : 2\cos\omega - 1 < 0\} \end{cases}$$

Thus, for $|\omega| < \pi$ we have

$$\angle H_d(\omega) = \begin{cases} -\omega & |\omega| < \dfrac{\pi}{3} \\ -\omega + \pi & \dfrac{\pi}{3} < |\omega| < \pi \end{cases}.$$

This is obviously not linear because $\angle H_d(\omega)$ takes jumps of π at $\omega = \pm\dfrac{\pi}{3}$.



Summary: In this example, $\angle H_d(\omega)$ is generalized linear phase but it is not linear phase.

Example

Changing the previous example to

$$\{h_n\} = \left\{\frac{1}{4}, -1, \frac{1}{4}\right\}$$

results in a filter that not only has generalized linear phase — it also has linear phase. Students are encouraged to work this out as an exercise.

Example

Determine whether $H_d(\omega)$ corresponding to

$$\{h_n\} = \{-1, 3, 1\}$$

has generalized linear phase.

Solution

At first it appears that this filter might have Type 2 generalized linear phase. However, this is not the case because the middle coefficient is nonzero. Let's examine $H_d(\omega)$:

$$H_d(\omega) = -1 + 3e^{-j\omega} + e^{-j2\omega}$$

$$= e^{-j\omega}\left(-e^{j\omega} + 3 + e^{-j\omega}\right)$$

$$= e^{-j\omega}\left(3 - j2\sin\omega\right)$$

Notice that this cannot be put in the form $e^{j\left(\frac{\pi}{2}-\omega M\right)} R(\omega)$ where $R(\omega)$ is real. The nonzero middle coefficient prevents this!

Example

Determine whether $H_d(\omega)$ corresponding to

$$\{h_n\} = \{1, -1\}$$

has generalized linear phase and linear phase.

<u>Solution</u>

$H_d(\omega)$ has Type 2 generalized linear phase since h_n is antisymmetric. Find the phase of $H_d(\omega)$ to check whether we have linear phase:

$$H_d(\omega) = 1 - e^{-j\omega}$$

$$= e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)$$

$$= e^{-j\frac{\omega}{2}}\, 2\, j \sin\frac{\omega}{2}$$

$$= e^{j\left(\frac{\pi}{2}-\frac{\omega}{2}\right)} \underbrace{2\sin\frac{\omega}{2}}_{R(\omega)}$$

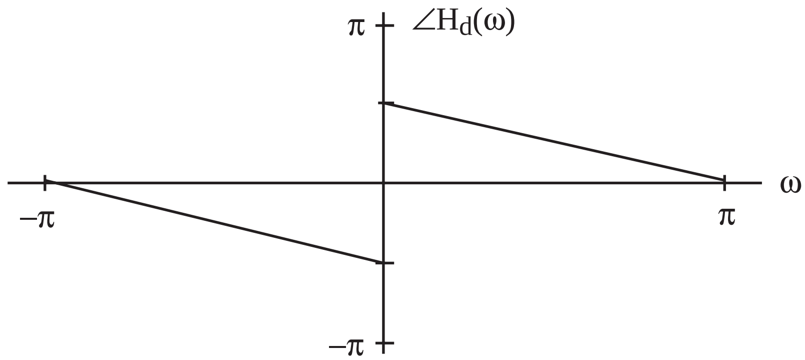
Here $R(\omega)$ is odd, so it must change sign at $\omega = 0$. This implies that we do not have linear phase. Let's find the phase:

$$\angle H_d(\omega) = \begin{cases} \frac{\pi}{2} - \frac{\omega}{2} & \left\{\omega : \sin\frac{\omega}{2} > 0\right\} \\ -\frac{\pi}{2} - \frac{\omega}{2} & \left\{\omega : \sin\frac{\omega}{2} < 0\right\} \end{cases}$$

Thus, for $|\omega| < \pi$ we have

$$\angle H_d(\omega) = \begin{cases} \dfrac{\pi}{2} - \dfrac{\omega}{2} & 0 < \omega < \pi \\ -\dfrac{\pi}{2} - \dfrac{\omega}{2} & -\pi < \omega < 0 \end{cases}$$

This is clearly not linear, as shown in the following plot.



Since $R(\omega)$ will be odd for any antisymmetric filter, we conclude that filters with antisymmetric coefficients cannot have linear phase.

Example

Given

$$h_n = \{\underset{\uparrow}{1}, 1\}$$

does $H_d(\omega)$ have generalized linear phase? How about linear phase?

Solution

Because of coefficient symmetry, $H_d(\omega)$ has Type 1 generalized linear phase. To check linear phase, look at:

$$\begin{aligned} H_d(\omega) &= 1 + e^{-j\omega} \\ &= e^{-j\frac{\omega}{2}} \left(e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}} \right) \\ &= e^{-j\frac{\omega}{2}} \underbrace{2\cos\frac{\omega}{2}}_{R(\omega)} \end{aligned}$$

Here $R(\omega)$ does not change sign on $-\pi < \omega < \pi$ and we have

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$$\angle H_d(\omega) = -\frac{\omega}{2} \qquad |\omega| \leq \pi$$

$\Rightarrow$ Strictly linear phase.

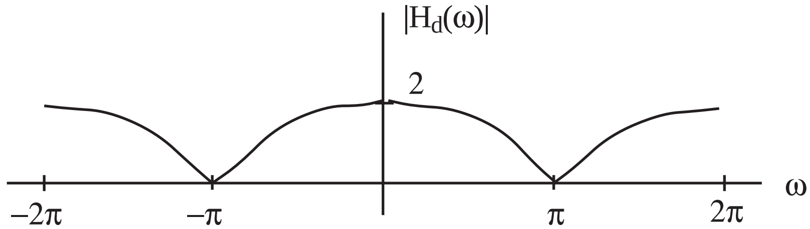
Of course, $\angle H_d(\omega)$ is periodic outside $|\omega| < \pi$.

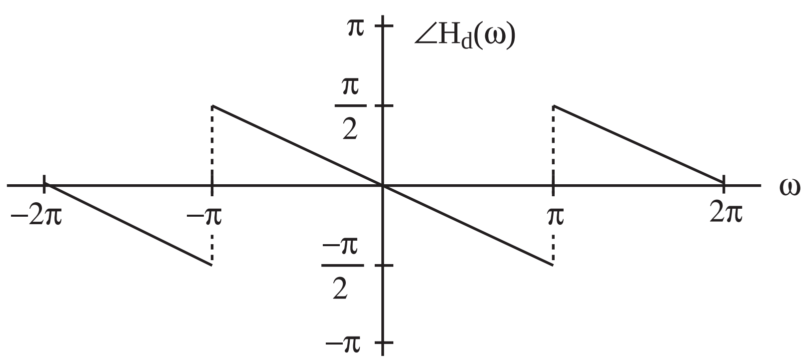
We have:

$$|H_d(\omega)| = 2\cos\frac{\omega}{2} \qquad |\omega| \leq \pi$$

$$\angle H_d(\omega) = -\frac{\omega}{2} \qquad |\omega| \leq \pi$$

So:





Here we do have jumps of π at ω = odd multiples of π, but we will still call this linear phase.

Impact of Coefficient Symmetry on Realizable Frequency Responses

Depending on whether $\{h_n\}_{n=0}^{N-1}$ are symmetric or antisymmetric, and N is even or odd, there can be restrictions on the types of filters that can be realized.

Example

If N is even (number of coefficients is even) and $\{h_n\}_{n=0}^{N-1}$ are symmetric, then you can't realize a high-pass filter! Why not? Because, for this case $H_d(\pi) = 0$, so that $\omega = \pi$ can't be in the passband for this type of filter. Let's show this.

For N even and h_n symmetric, we have

$$H(z) \quad = \quad h_0 + h_1 z^{-1} + h_2 z^{-2} + \ldots + h_2 z^{-(N-3)} + h_1 z^{-(N-2)} + h_0 z^{-(N-1)} .$$

Then

$$H_d(\pi) \quad = \quad H(-1) = h_0 - h_1 + h_2 - \ldots - h_2 + h_1 - h_0 = 0$$

In practice, it pays to be aware of these types of constraints, but the problem is easily resolved. For example, in designing a high-pass filter with symmetric coefficients, we would simply take N to be odd.

Let's now address this problem in more generality by considering some short FIR filters to see what restrictions exist on $H_d(0)$ and $H_d(\pi)$ as a function of coefficient symmetry and the value of N.

$$H_d(\omega) \quad = \quad a_0 + a_1 e^{-j\omega} + a_1 e^{-j2\omega} + a_0 e^{-j3\omega} \qquad \text{(even symmetry, N even)}$$

$$\Rightarrow \begin{cases} H_d(0) = 2a_0 + 2a_1 \\ H_d(\pi) = 0 \end{cases}$$

$$H_d(\omega) \quad = \quad a_0 + a_1 e^{-j\omega} + a_2 e^{-j2\omega} + a_1 e^{-j3\omega} + a_0 e^{-j4\omega} \quad \text{(even symmetry, N odd)}$$

$$\Rightarrow \begin{cases} H_d(0) = 2a_0 + 2a_1 + a_2 \\ H_d(\pi) = 2a_0 - 2a_1 + a_2 \end{cases}$$

$$H_d(\omega) \quad = \quad a_0 + a_1 e^{-j\omega} - a_1 e^{-j2\omega} - a_0 e^{-j3\omega} \qquad \text{(odd symmetry, N even)}$$

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$$\Rightarrow \begin{cases} H_d(0) = 0 \\ H_d(\pi) = 2a_0 - 2a_1 \end{cases}$$

$H_d(\omega) \;=\; a_0 + a_1\, e^{-j\omega} + 0\, e^{-j2\omega} - a_1\, e^{-j3\omega} - a_0\, e^{-j4\omega}$ (odd symmetry, N odd)

$$\Rightarrow \begin{cases} H_d(0) = 0 \\ H_d(\pi) = 0 \end{cases}$$

In general, we conclude

Symmetry	**N**	**Unrealizable Filters**
even	even	high-pass, bandstop
even	odd	no restriction
odd	even	low-pass, bandstop
odd	odd	low-pass, high-pass, bandstop

Notice that a bandstop filter has its stopband located between 0 and π and therefore has passbands centered at both $\omega = 0$ and $\omega = \pi$. Thus, if either a lowpass or highpass filter cannot be realized, this implies that a bandstop filter cannot be realized.