ECE 410   **DIGITAL SIGNAL PROCESSING**   **D. Munson**
University of Illinois   **Chapter 11**

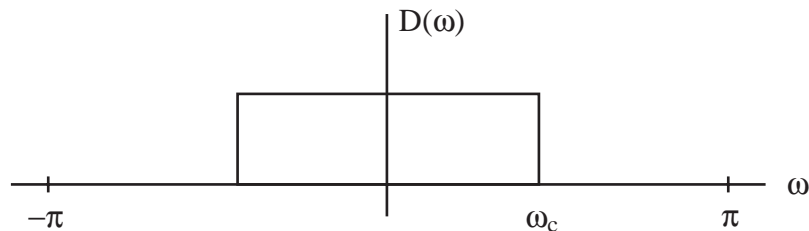## DIGITAL FILTER DESIGN

FIR Design Methods

1) Windowing

    a) Truncation
    b) General windowing

2) Frequency Sampling ~ will cover later in connection with the DFT

3) Computer-Aided Optimization

    a) Parks - McClellan ($\sqrt{}$) ~ widely used
    b) Linear Programming

1a) Truncation

Illustrate by example.
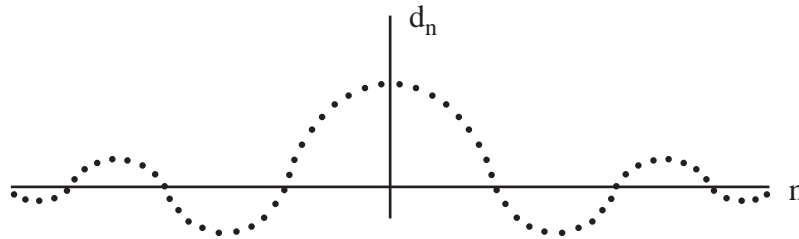
Suppose we want a LPF with frequency response:

We might choose the filter coefficients $\{h_n\}$ to be

$$d_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} D(\omega)\, e^{j\omega n}\, d\omega \;=\; \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1\, e^{j\omega n}\, d\omega = \frac{\omega_c}{\pi}\, \text{sinc}\, \omega_c\, n$$
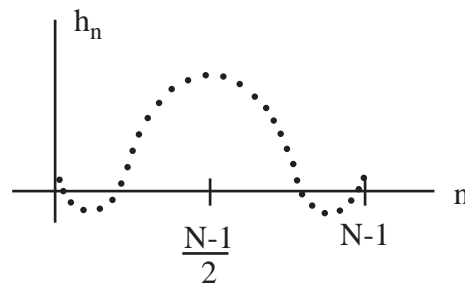
(Then frequency response $H_d(\omega) = \text{DTFT}\left[\{d_n\}\right] = D(\omega)$ exactly.)

Look at the d_n:

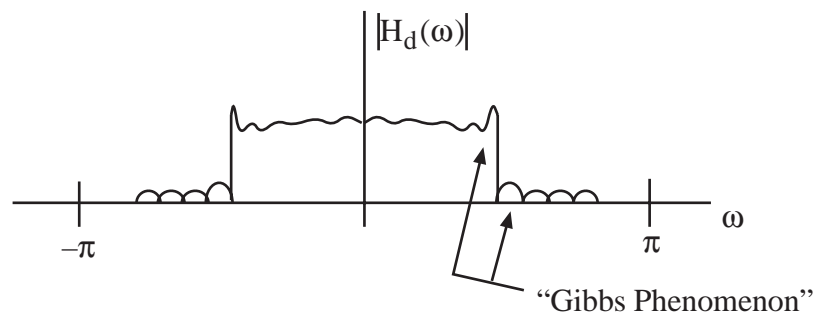This sequence is infinite in length and noncausal. So, let's choose (assume N odd, for now):

$$h_n = \begin{cases} d_{n-\frac{N-1}{2}} & 0 \le n \le N-1 \\ 0 & \text{else} \end{cases}$$

If N is large, then h_n consists of all the larger values of d_n and

$$H_d(\omega) \approx D(\omega)\, e^{-j\omega\frac{N-1}{2}}.$$

But this approximation is poor in the following sense. The frequency response will look like:

"Gibbs Phenomenon"

The ripples are due to the Gibbs' phenomenon. Get tall ripples at sharp transitions. As N increases, ripples become narrower and more numerous, but the heights of the ripples nearest the discontinuity remain large. We will soon develop our own explanation for why this occurs. For now, it is worth noting that the DTFT

$$\sum_{n=-\infty}^{\infty} d_n \, e^{-j\omega n}$$

is actually a Fourier series expansion of $D(\omega)$. We had not thought of the DTFT in this way before, because there was no advantage in doing so. Here, though, we note that $H_d(\omega)$ is obtained by <u>truncating</u> this Fourier series (i.e., choosing $h_n = d_{n-\frac{N-1}{2}}$ for $0 \le n \le N - 1$.)

Gibbs and other mathematicians studied truncation of Fourier series and showed that ripples will occur around locations where the periodic function is discontinuous. These ripples can be made narrower and to bunch up around the points of discontinuity by taking N larger. However, a larger N does not reduce the heights of the ripples!

How can we reduce the ripple heights?

1b) General Windowing

Given $D(\omega)$ and the corresponding infinite-length $\{d_n\}$, choose the coefficients $\{h_n\}_{n=0}^{N-1}$ to be (again, assume N odd for now):

$$h_n = w_n \, d_{n-\frac{N-1}{2}} \qquad 0 \le n \le N - 1$$

where $w_n = 0$, $n \notin [0, N-1]$, is a <u>window</u> sequence that gently tapers to zero.

For truncation we used

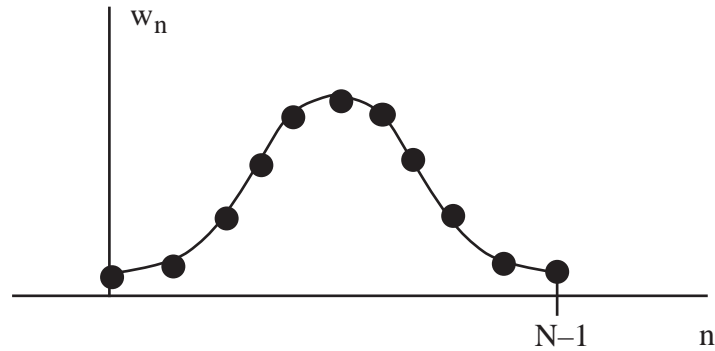$$w_n = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{else} \end{cases}$$

which does *not* taper gently to zero. Smoother windows can lead to <u>much lower ripple</u>.

An example of a good window is the Hamming window:

$$w_n = .54 - .46 \cos\left(\frac{2\pi n}{N-1}\right), \quad 0 \le n \le N - 1.$$

This is plotted below

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How does the window choice affect the frequency response of the designed filter $\{h_n\}_{n=0}^{N-1}$?

Let

$$g_n = d_{n-\frac{N-1}{2}} \ .$$

Then:

$$h_n = w_n \, g_n$$

desired u.p.r. after shifting
window
coeffs. of designed FIR filter

In the frequency domain this corresponds to:

$$H_d(\omega) \;=\; \sum_n w_n \, g_n \, e^{-j\omega n}$$

$$=\; \sum_n w_n \, \frac{1}{2\pi} \int_{-\pi}^{\pi} G_d(\theta) e^{j\theta n} \, d\theta \; e^{-j\omega n}$$

$$=\; \frac{1}{2\pi} \int_{-\pi}^{\pi} G_d(\theta) \sum_n w_n \, e^{-jn(\omega - \theta)} \, d\theta$$

$$=\; \frac{1}{2\pi} \int_{-\pi}^{\pi} G_d(\theta) \, W_d(\omega - \theta) \, d\theta \qquad (*)$$

Notice that this has the form of a convolution. Since the integrand is periodic and we integrate over only a single period, this is called a periodic convolution.

Since our goal is to have $H_d(\omega) \approx G_d(\omega)$, we see from ($*$) that we would like

$$W_d(\omega) = 2\pi\, \delta(\omega)$$

But, $W_d(\omega) = 2\pi\, \delta(\omega) \Rightarrow$

$$w_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi \delta(\omega)\, e^{j\omega n}\, d\omega = 1 \qquad \text{for all n}$$

$\Rightarrow w_n$ is not a window sequence!

w_n nonzero only on $n \; \varepsilon \; [0, N-1]$ results in (at best) either:

   i) $W_d(\omega)$ is a narrow pulse around $\omega = 0$, but has high sidelobes.

or

   ii) $W_d(\omega)$ is wider around the origin but the sidelobes are lower.

     So, we have a <u>tradeoff</u>.

From ($*$) we see that the ripple in $H_d(\omega)$ is caused by integrating the product of G_d and W_d as the ripples in W_d are shifted across the discontinuity of G_d. Likewise, the width of the transition band for H_d will depend on the width of the mainlobe (center lobe) of W_d. We conclude:

High sidelobes of $W_d \Rightarrow$ Large ripple in H_d.
Wide center lobe of $W_d \Rightarrow$ Wide transition band in H_d.

Our goal is to achieve moderately low sidelobes and a moderately narrow transition width in W_d <u>simultaneously</u>.
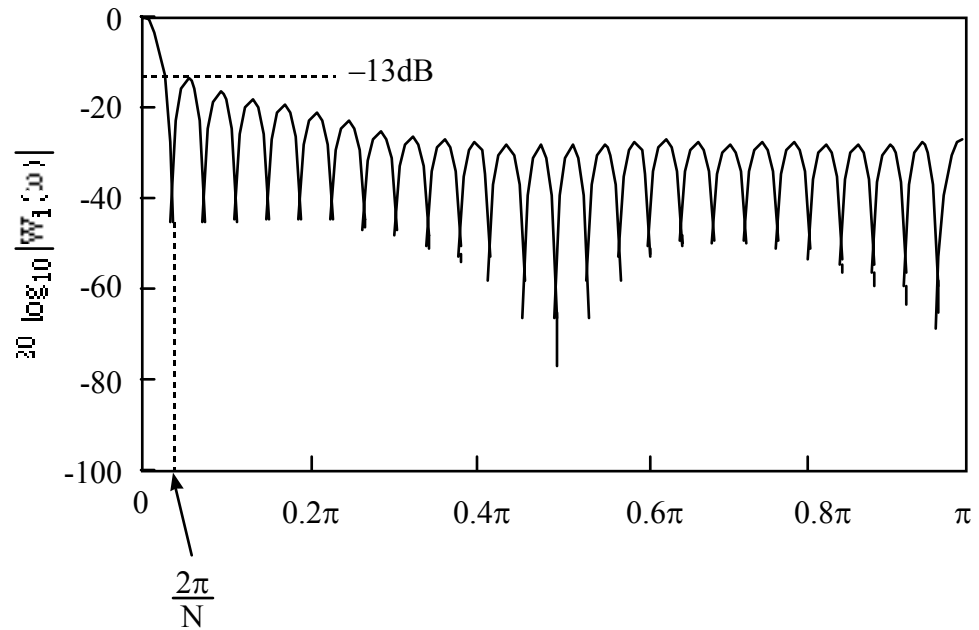
What does $W_d(\omega)$ look like for some common windows?
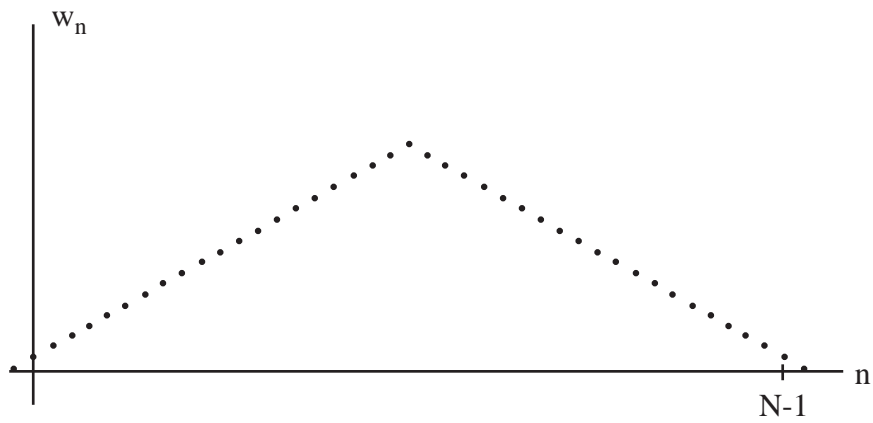
<u>Truncation</u>

$$\left|W_d(\omega)\right| = \left|\sum_{n=0}^{N-1} 1\, e^{-j\omega n}\right| = \left|\frac{\sin\dfrac{N}{2}\omega}{\sin\dfrac{1}{2}\omega}\right|$$
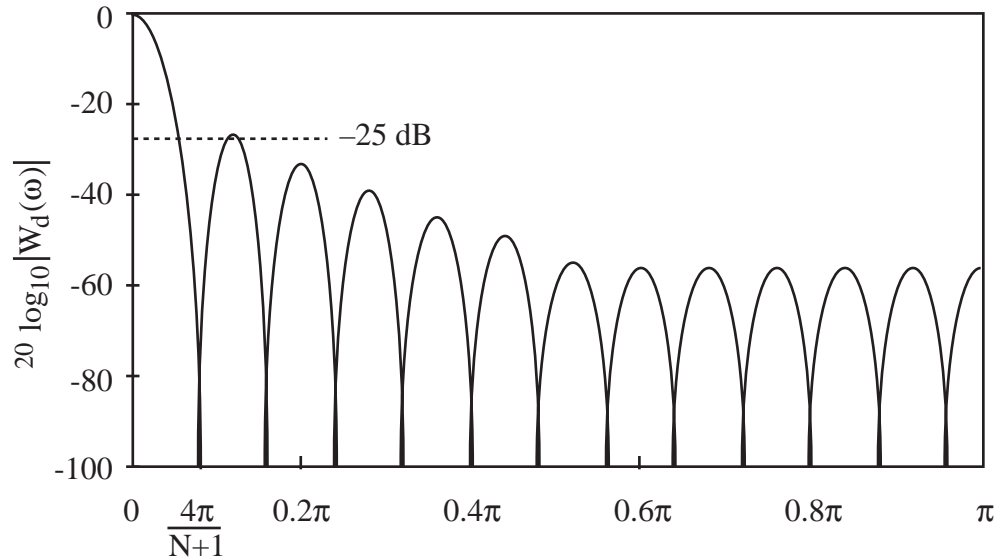
11.6

Plot on log scale:

Triangular (Bartlett):

$$|W_d(\omega)| = \left| \frac{\sin \dfrac{N+1}{4}\omega}{\sin \dfrac{1}{2}\omega} \right|^2$$

Plot on log scale:

From this plot, we see that the triangular window has a mainlobe that is twice as wide as that of the rectangular window, but the highest sidelobe is reduced by 12 db.

Hamming:

$$w_n = .54 - .46 \cos\left(\frac{2\pi n}{N-1}\right) \quad 0 \le n \le N-1$$

Plot of $\left|W_d(\omega)\right|$ on a log scale:
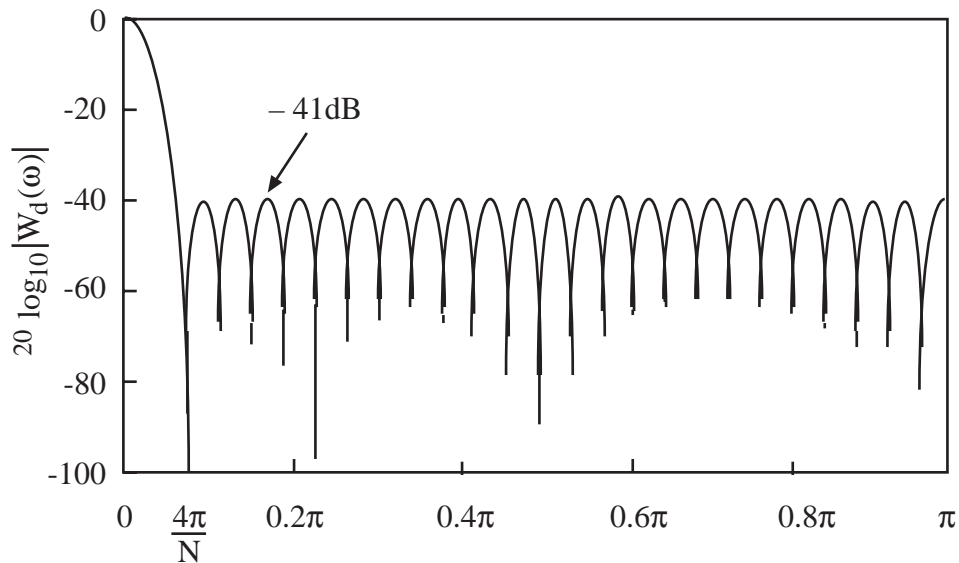
From this plot, we see that the Hamming window has essentially the same mainlobe width as the triangular window, and sidelobes that are reduced an additional 16 dB to –41 dB. Thus, the

Hamming window is preferred over the triangular window. Comparing the Hamming window to the truncation window, we see that the highest sidelobe is reduced by 28 dB (more than a factor of 10) at the expense of increasing the mainlobe width by a factor of 2.

Best window: <u>Kaiser</u>

$$w_n = I_0\left[\beta\left(1 - \left[\left(n - \frac{N-1}{2}\right)\bigg/\frac{N-1}{2}\right]^2\right)^{1/2}\right] \qquad 0 \le n \le N-1$$

I_0 is the zeroth-order modified Bessel function of the first kind:

$$I_0(x) = \frac{1}{\pi}\int_0^\pi e^{\pm x\cos\theta}\, d\theta = \frac{1}{\pi}\int_0^\pi \cosh(x\cos\theta)\, d\theta$$

The choice of β affects the tradeoff between the mainlobe width and sidelobe heights. β is user specified.

The Kaiser window can achieve slightly narrower mainlobe with the same sidelobe height as Hamming window.
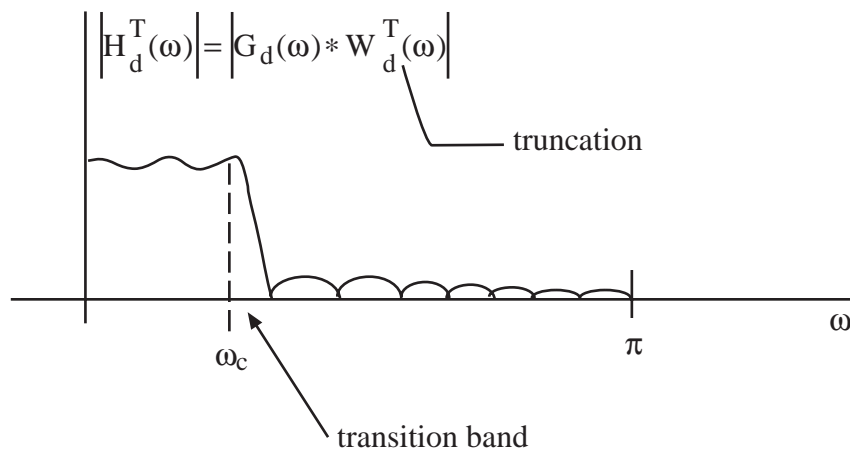
Typical Frequency Responses Using Window Design

If the desired magnitude response is

$$\left|G_d(\omega)\right|$$

$$\omega_c \qquad \pi \qquad \omega$$

then the frequency response of a truncation design may look like:

$$\left|\left|H_d^T(\omega)\right|\right| = \left|G_d(\omega) * W_d^T(\omega)\right|$$

truncation

$$\omega_c \qquad \pi \qquad \omega$$

transition band

and the frequency response of a Hamming design will look like:

$$\left|\left|H_d^H(\omega)\right|\right| = \left|G_d(\omega) * W_d^H(\omega)\right|$$

Hamming

$$\omega_c \qquad \pi \qquad \omega$$
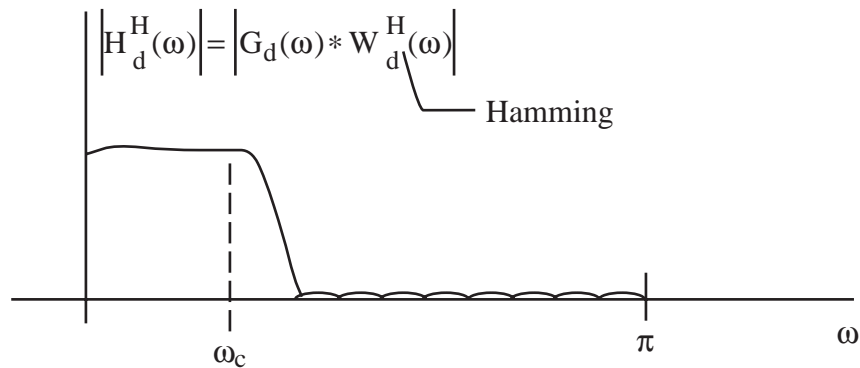
So, Hamming widens the transition band by a factor of two, but greatly reduces ripple.

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<u>Note:</u>   The actual filter design procedure is

$$h_n = w_n\, g_n \qquad 0 \le n \le N - 1.$$

The above Fourier-domain concepts are to help us visualize the resulting $H_d(\omega)$.

Now, so far we have considered only the case with N odd, where we defined

$$g_n = d_{n-\frac{N-1}{2}}$$

where $D(\omega)$ is the desired $H_d(\omega)$.

How do we find g_n if N is even? Answer: Select g_n as shown in the following procedure, which works for both N odd and N even.

<u>General Window Design Procedure</u>

To design a generalized linear phase $\{h_n\}_{n=0}^{N-1}$ with $|H_d(\omega)| \approx D(\omega)$ do this:

1) Let $G_d(\omega) = D(\omega) e^{-j\frac{N-1}{2}\omega}$

2) Find $g_n = \text{DTFT}^{-1}\left[G_d(\omega)\right]$

3) Let $h_n = w_n\, g_n$.

<u>Notes</u>:

       1) For N odd this procedure gives $g_n = d_{n-\frac{N-1}{2}}$ as before. For N even, steps 1) and

      2) give $\{g_n\}$ as an interpolated set of values lying between $\left\{d_{n-\frac{N}{2}}\right\}$ and $\left\{d_{n-\frac{N-2}{2}}\right\}$.

       2) We wish to know whether $H_d(\omega)$, designed via the window method, will have generalized linear phase. The answer is ordinarily yes, since $\{h_n\}$ will be symmetric or antisymmetric if $\{w_n\}$ is symmetric and $\{g_n\}$ is either symmetric or antisymmetric.

FIR Window Design Examples

Example

Design generalized linear-phase, low-pass FIR filters having coefficients $\{h_n\}_{n=0}^{29}$ and cutoff

$\omega_c = \frac{\pi}{4}$ using the window design procedure with both truncation and Hamming windows.

Solution

$$G_d(\omega) \;=\; D(\omega)e^{-j\frac{N-1}{2}\omega}$$

$$= \begin{cases} e^{-j\frac{29}{2}\omega} & |\omega| \le \dfrac{\pi}{4} \\[2mm] 0 & \dfrac{\pi}{4} < |\omega| \le \pi \end{cases}$$

$$\Rightarrow \quad g_n \;=\; \frac{1}{2\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{-j\frac{29}{2}\omega}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \left. \frac{e^{j\omega\left(n-\frac{29}{2}\right)}}{j\left(n-\frac{29}{2}\right)} \right|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{\sin\dfrac{\pi}{4}\left(n-\dfrac{29}{2}\right)}{\pi\left(n-\dfrac{29}{2}\right)}$$

$$= \frac{1}{4}\,\mathrm{sinc}\left[\frac{\pi}{4}\left(n-\frac{29}{2}\right)\right]$$

Now, for truncation $h_n \;=\; w_n\, g_n$ with $w_n = \begin{cases} 1 & 0 \le n \le 29 \\ 0 & \text{else} \end{cases}$.

So:

$$\boxed{\; h_n \;=\; \frac{1}{4}\,\mathrm{sinc}\,\frac{\pi}{4}\left(n-\frac{29}{2}\right) \qquad 0 \le n \le 29 \;}$$

For the Hamming window design we have

$$h_n = \left[.54 - .46 \cos\frac{2\pi n}{29} \right] \frac{1}{4} \operatorname{sinc}\frac{\pi}{4}\left(n - \frac{29}{2} \right) \quad 0 \le n \le 29$$

Example

Design generalized linear phase high-pass FIR filters having coefficients $\{h_n\}_{n=0}^{60}$ and cutoff $\omega_c = \frac{2\pi}{3}$ using the window design procedure with both truncation and Hamming windows.

<u>Solution</u>

$$G_d(\omega) = \begin{cases} e^{-j30\omega} & \frac{2\pi}{3} \le |\omega| \le \pi \\ \\ 0 & |\omega| < \frac{2\pi}{3} \end{cases}$$

$$\Rightarrow g_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} G_d(\omega) e^{j\omega n} \, d\omega$$

$$\underbrace{\qquad\qquad}_{\text{periodic with period} = 2\pi}$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} G_d(\omega) e^{j\omega n} \, d\omega \qquad (*)$$

$$= \frac{1}{2\pi} \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} e^{-j30\omega} e^{j\omega n} \, d\omega \qquad (\square)$$

$$= \frac{e^{j\omega(n-30)}}{2\pi j(n-30)} \Bigg|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \qquad (\Delta)$$

$$= \frac{1}{2\pi j(n-30)} \left[e^{j\frac{4\pi}{3}(n-30)} - e^{j\frac{2\pi}{3}(n-30)} \right]$$

$$= \frac{1}{2\pi j(n-30)} e^{j\pi(n-30)} \left[e^{j\frac{\pi}{3}(n-30)} - e^{-j\frac{\pi}{3}(n-30)} \right]$$

$$= \frac{1}{2\pi j(n-30)} (-1)^n \, 2j \sin\frac{\pi}{3}(n-30)$$

$$= (-1)^n \frac{1}{3} \, \text{sinc} \, \frac{\pi}{3}(n-30)$$

So, using a truncation window gives:

$$\boxed{h_n = (-1)^n \frac{1}{3}\text{sinc}\,\frac{\pi}{3}(n-30) \quad 0 \le n \le 60}$$

and applying a Hamming window gives:

$$\boxed{h_n = \left[.54 - .46\cos\frac{2\pi n}{60}\right](-1)^n\frac{1}{3}\text{sinc}\,\frac{\pi}{3}(n-30) \quad 0 \le n \le 60}$$

Notes:

1) Since $G_d(\omega)$ is nonzero on <u>two</u> subintervals of $-\pi \le \omega < \pi$ for the high-pass case, it can save algebra if we use periodicity to rewrite the inverse DTFT as in (*) across the interval $0 \le \omega \le 2\pi$. This trick is straightforward if N is odd. For N even, however, there are two extra things to think about. First, in view of the table on p. 35.8, we should use an odd-symmetric design with Type 2 generalized linear phase. Second, for N even, the slope of the phase, $-\frac{N-1}{2}$, is noninteger and the phase will take a jump of π at $\omega = \pi$. Thus, $G_d(\omega)$ seemingly will have two different forms on the interval $0 \le \omega \le 2\pi$. See the next example for details.

2) Since the denominator of (Δ) is zero at n = 30, we cannot presume that (Δ) follows from ($\square$) at n = 30. Thus, we must be careful to check that our expressions for $\{h_n\}$ hold at n = 30. Our final expression for g_n, which follows from (Δ), gives:

$$g_{30} = (-1)^{30} \cdot \frac{1}{3} \cdot 1 = \frac{1}{3}$$

Evaluating ($\square$) at n = 30 gives

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$$g_{30} = \frac{1}{2\pi}\left[\frac{4\pi}{3} - \frac{2\pi}{3}\right] \cdot 1 = \frac{1}{3}$$

which agrees with (Δ). Thus, our expressions for h_n are valid for $0 \le n \le 60$.

Now, let's change N in the previous example from $N = 61$ to $N = 62$ and see how the algebra associated with the design changes for N even.

Example

Design a generalized linear phase high-pass FIR filter having coefficients $\{h_n\}_{n=0}^{61}$ and cutoff $\omega_c = \frac{2\pi}{3}$ using the window design procedure with a Hamming window.

Solution

A filter with real-valued unit-pulse response satisfies $G_d(\omega) = G_d^*(-\omega)$. Thus, for an antisymmetric design with Type 2 generalized linear phase, we have
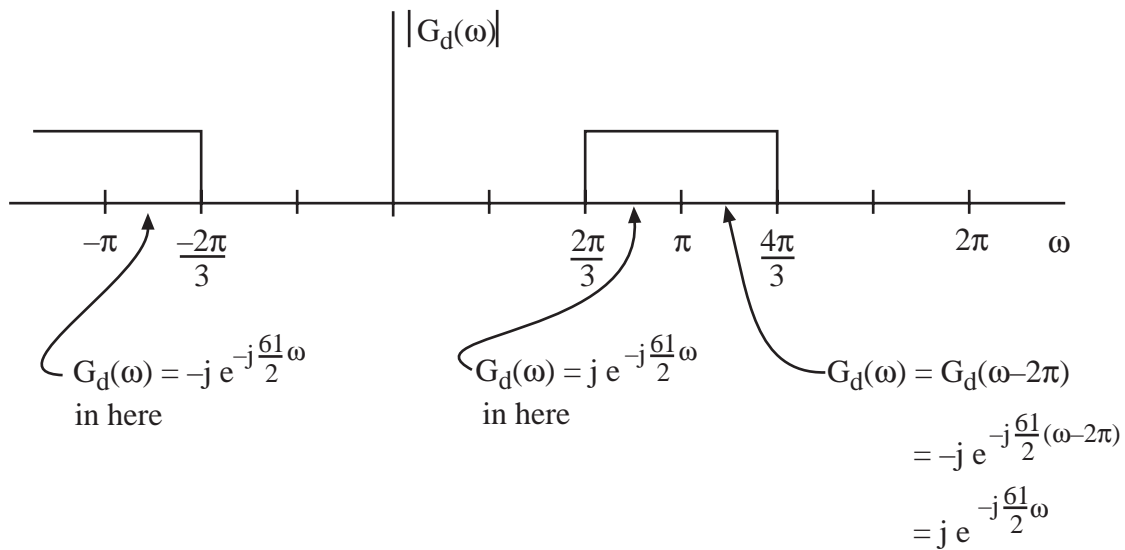
$$G_d(\omega) = \begin{cases} e^{j\left(\frac{\pi}{2} - \frac{61}{2}\omega\right)} & \frac{2\pi}{3} \le \omega < \pi \\ 0 & |\omega| < \frac{2\pi}{3} \\ e^{j\left(-\frac{\pi}{2} - \frac{61}{2}\omega\right)} & -\pi < \omega \le -\frac{2\pi}{3} \end{cases}$$

$$= \begin{cases} j\, e^{-j\frac{61}{2}\omega} & \frac{2\pi}{3} \le \omega < \pi \\ 0 & |\omega| < \frac{2\pi}{3} \\ -j\, e^{-j\frac{61}{2}\omega} & -\pi < \omega \le -\frac{2\pi}{3} \end{cases} \qquad (*)$$

As in the last example we can write

$$g_n = \frac{1}{2\pi} \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} G_d(\omega)\, e^{j\omega n}\, d\omega \qquad\qquad (**)$$

Equation (*) specifies $G_d(\omega)$ on $|\omega| < \pi$. To find $G_d(\omega)$ for $\pi < \omega < \frac{4\pi}{3}$ consider

$$G_d(\omega) = -j\, e^{-j\frac{61}{2}\omega}$$
in here

$$G_d(\omega) = j\, e^{-j\frac{61}{2}\omega}$$
in here

$$G_d(\omega) = G_d(\omega - 2\pi)$$
$$= -j\, e^{-j\frac{61}{2}(\omega - 2\pi)}$$
$$= j\, e^{-j\frac{61}{2}\omega}$$

So, $G_d(\omega)$ in (**) maintains the same form across the full range of integration in (**) and we have

$$g_n = \frac{1}{2\pi} \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} j e^{-j\frac{61}{2}\omega}\, e^{j\omega n}\, d\omega$$

$$= \left. \frac{e^{j\omega\left(n - \frac{61}{2}\right)}}{2\pi\left(n - \frac{61}{2}\right)} \right|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}$$

$$= \frac{1}{2\pi\left(n - \frac{61}{2}\right)} \left[e^{j\frac{4\pi}{3}\left(n - \frac{61}{2}\right)} - e^{j\frac{2\pi}{3}\left(n - \frac{61}{2}\right)} \right]$$
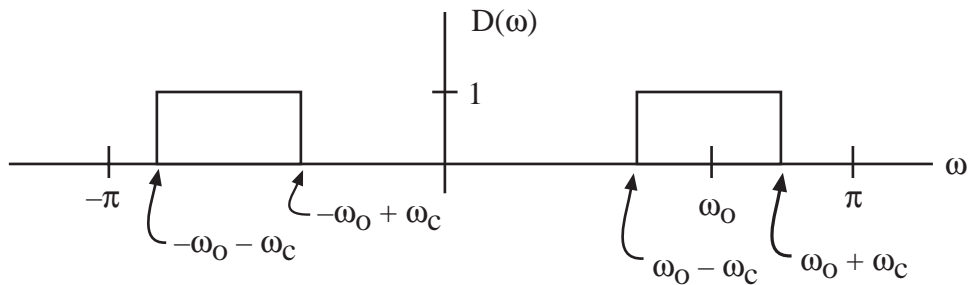
$$= \frac{1}{2\pi\left(n - \frac{61}{2}\right)} e^{j\pi\left(n - \frac{61}{2}\right)} \left[e^{j\frac{\pi}{3}\left(n - \frac{61}{2}\right)} - e^{-j\frac{\pi}{3}\left(n - \frac{61}{2}\right)} \right]$$

$$= \frac{1}{2\pi\left(n - \frac{61}{2}\right)} (-1)^n \, 2\sin\left[\frac{\pi}{3}\left(n - \frac{61}{2}\right)\right]$$

$$= (-1)^n \, \frac{1}{3} \, \text{sinc} \, \frac{\pi}{3}\left(n - \frac{61}{2}\right)$$

Applying the Hamming window gives

$$\boxed{h_n = \left[.54 - .46\cos\frac{2\pi n}{61}\right](-1)^n \frac{1}{3}\,\text{sinc}\,\frac{\pi}{3}\left(n - \frac{61}{2}\right) \qquad 0 \le n \le 61}$$

Example

Window design of bandpass filter. Find $\{h_n\}_{n=0}^{N-1}$ so that $|H_d(\omega)|$ approximates

For lowpass filter had $d_n = \frac{\omega_c}{\pi}\,\text{sinc}\,\omega_c n$. By modulation property, here for BP case we expect

$d_n \approx (\cos\omega_o n)\frac{\omega_c}{\pi}\,\text{sinc}\,[\omega_c n]$. Let's see:

$$d_n = \frac{1}{2\pi}\int_{-\omega_o - \omega_c}^{-\omega_o + \omega_c} 1\,e^{j\omega n}\,d\omega + \frac{1}{2\pi}\int_{\omega_o - \omega_c}^{\omega_o + \omega_c} 1\,e^{j\omega n}\,d\omega$$

$$= \frac{e^{j\omega n}}{2\pi j n}\bigg|_{-\omega_o - \omega_c}^{-\omega_o + \omega_c} + \frac{e^{j\omega n}}{2\pi j n}\bigg|_{\omega_o - \omega_c}^{\omega_o + \omega_c}$$

$$= \frac{e^{jn(-\omega_o + \omega_c)} - e^{jn(-\omega_o - \omega_c)} + e^{jn(\omega_o + \omega_c)} - e^{jn(\omega_o - \omega_c)}}{2\pi jn}$$

$$= \frac{1}{2\pi jn}\left[e^{-jn\omega_o}\left(e^{jn\omega_c} - e^{-jn\omega_c}\right) + e^{jn\omega_o}\left(e^{jn\omega_c} - e^{-jn\omega_c}\right)\right]$$

$$= \frac{1}{\pi n}\left[e^{-jn\omega_o}\sin\omega_c n + e^{jn\omega_o}\sin\omega_c n\right] = \frac{\sin\omega_c n}{\pi n} 2\cos\omega_o n$$

$$= 2\left(\cos\omega_o n\right)\frac{\omega_c}{\pi}\text{sinc }\omega_c n \qquad \leftarrow \text{what we expected.}$$

Now, to design $\{h_n\}_{n=0}^{N-1}$ with linear phase, need to incorporate shift to give:

$$\boxed{g_n = 2\cos\omega_o\left(n - \frac{N-1}{2}\right)\frac{\omega_c}{\pi}\text{sinc }\omega_c\left(n - \frac{N-1}{2}\right) \qquad 0 \le n \le N-1}$$

Windowed coefficients are then

$\boxed{h_n = w_n g_n \qquad 0 \le n \le N-1}$ ~ coefficients for FIR filter where w_n is a Hamming or other window.

In MATLAB, the command to design FIR filters using the window method is called `fir1`.

Parks-McClellan

Parks and McClellan developed a computer program for solving the following problem:
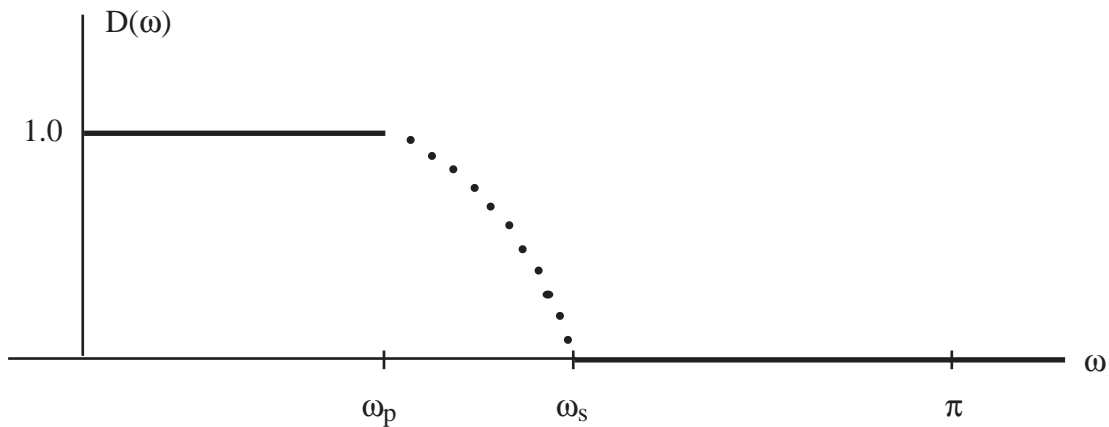
Define the error

$$E(\omega) = W(\omega)[D(\omega) - R(\omega)]$$

arbitrary weighting

desired response

from

$$H_d(\omega) = R(\omega)e^{j\left(\alpha - \frac{N-1}{2}\omega\right)}$$

$$0 \text{ or } \frac{\pi}{2}$$

Let ω_p, ω_s be the passband and stopband cutoff frequencies so that $D(\omega)$ might look like:

Ordinarily $D(\omega)$ is unspecified on $\omega_p < \omega < \omega_s$ because we are not concerned with the precise shape of $H_d(\omega)$ or $R(\omega)$ in this <u>transition band</u>.

The Parks-McClellan algorithm finds $\{h_n\}_{n=0}^{N-1}$ that minimizes

$$\max_{\substack{0 \le \omega \le \omega_p \\ \omega_s \le \omega \le \pi}} |E(\omega)|$$

This error measure is called the minimax or Chebyshev error norm.

In the standard P-M algorithm, $W(\omega)$ can be selected to have one value, W_p, on the passband and another, W_s, on the stopband. Frequently, W_s is chosen larger than W_p so that the designed filter will have a smaller stopband error than passband error.

The program user specifies:

N, ω_p, ω_s, W_p, and W_s

The designed filter has <u>equiripple</u> behavior:
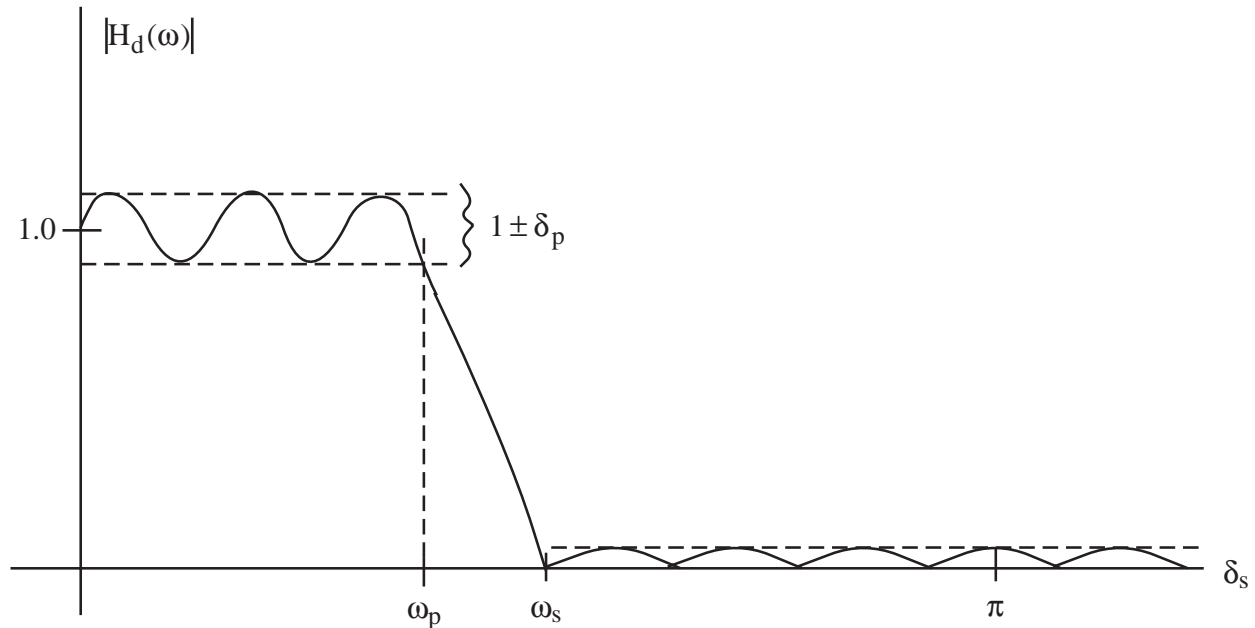
How are the ripple heights related to W_p and W_s?

Answer: They satisfy $\delta_p W_p = \delta_s W_s$.

Thus, $\dfrac{\delta_p}{\delta_s} = \dfrac{W_s}{W_p}$.

What filter order is required to meet given specifications?

Answer: It has been found empirically that

$$N \approx \frac{-10\log_{10}\left(\delta_p \delta_s\right) - 13}{2.324\left(\omega_s - \omega_p\right)}$$

Note: The filter order is not too sensitive to δ_p, and δ_s. <u>But</u>, N is inversely proportional to the transition bandwidth! Halving the transition bandwidth doubles the required filter length!

All students will get experience with the Parks-McClellan algorithm in a Matlab assignment.

Linear Programming

Linear programming is a fairly general optimization algorithm that can solve the Parks-McClellan problem and many others. LP is slower, but its generality can be exploited to incorporate time-domain or additional frequency-domain constraints.

For example, it is possible to eliminate ripple in the passband by constraining the derivative of $R(\omega)$ to be ≤ 0 in the passband.

The result is a monotone decreasing response in the passband, with ripple remaining in the stopband:
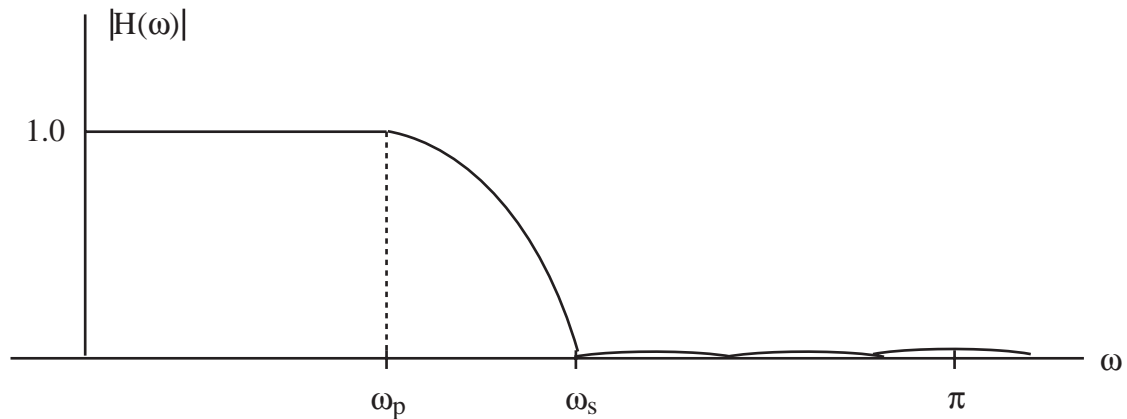
Incorporating the monotone passband constraint increases the required filter order slightly.

In MATLAB, the command to design FIR filters using the P-M method is called `remez`.

Frequency Sampling FIR Filter Design

Given $G_d(\omega) \sim$ desired frequency response with linear phase

Idea: Choose $H_d(\omega)$ to agree with $G_d(\omega)$ at $\omega = \dfrac{2\pi}{N} m$, $0 \leq m \leq N-1$.

Take $\{h_n\}_{n=0}^{N-1}$ to be $\text{DFT}^{-1}\left[\left\{G_d\left(\dfrac{2\pi m}{N}\right)\right\}_{m=0}^{N-1}\right]$, i.e.,

$$h_n = \frac{1}{N} \sum_{m=0}^{N-1} G_d\left(\frac{2\pi}{N} m\right) e^{j\frac{2\pi}{N} nm} \qquad 0 \leq n \leq N-1$$

(FS)

Using $\{h_n\}_{n=0}^{N-1}$ as our FIR filter coefficients will give a frequency response $H_d(\omega)$ satisfying

$$H_d(\omega) = G_d(\omega) \text{ at } \omega = \frac{2\pi m}{N} \qquad 0 \le m \le N - 1$$
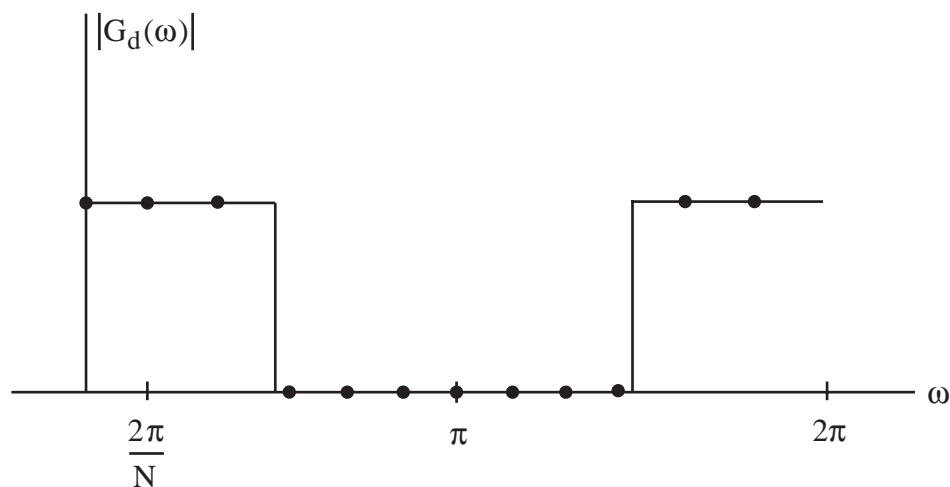
$\uparrow$ actual $\uparrow$ desired

Why?

$$H_d(\omega)\Big|_{\omega = \frac{2\pi}{N}m} = \text{DFT}\left[\{h_n\}\right]$$

$$= \text{DFT}\left[\text{DFT}^{-1}\left[\left\{G_d\left(\frac{2\pi}{N}m\right)\right\}\right]\right]$$

$\uparrow$ by (FS)

$$= G_d\left(\frac{2\pi}{N}m\right) \qquad \checkmark$$

<u>But</u> in general $H_d(\omega) \ne G_d(\omega)$ for $\omega \ne \frac{2\pi}{N}$ m.
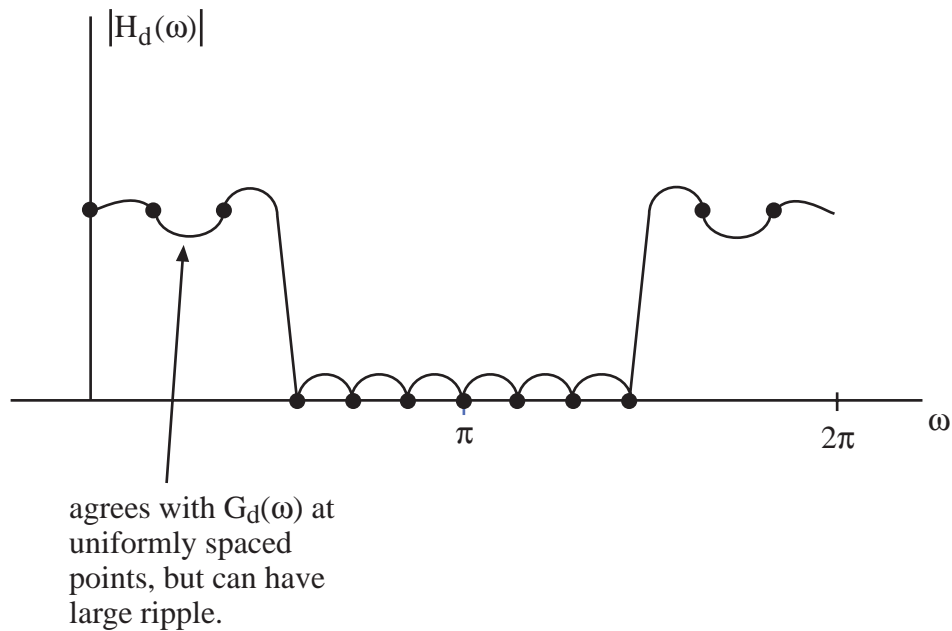
This approach <u>can be</u> too simplistic.

<u>Example</u>

Let $G_d(\omega)$ be an ideal LPF with linear phase and delay $\frac{N-1}{2}$.

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Using (FS), will produce an $H_d(\omega)$ that agrees with $G_d(\omega)$ at the points shown above. The resulting $H_d(\omega)$ might look like:

agrees with $G_d(\omega)$ at
uniformly spaced
points, but can have
large ripple.

There is a modified frequency sampling design procedure that works better. In the modified procedure, one or more frequency samples in the filter's transition band are left unconstrained as free variables. The values of these free variables are then selected via linear programming to minimize some measure of ripple. This results in a wider transition band, but lower ripple, as suggested in the figures below.

variable transition sample
that can be selected to
minimize ripple.

wider transition
band, lower ripple.
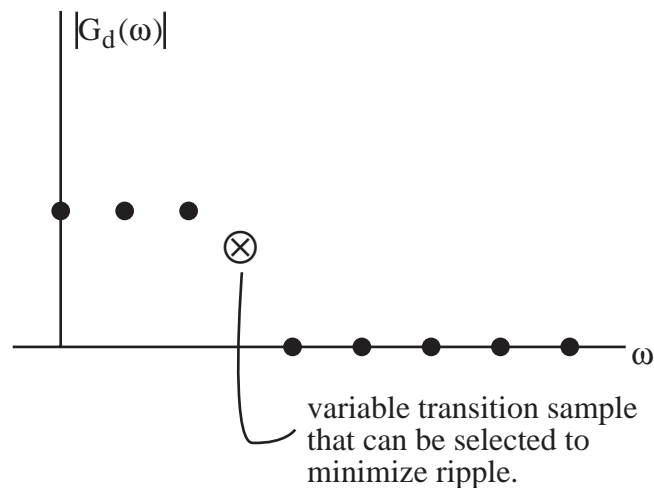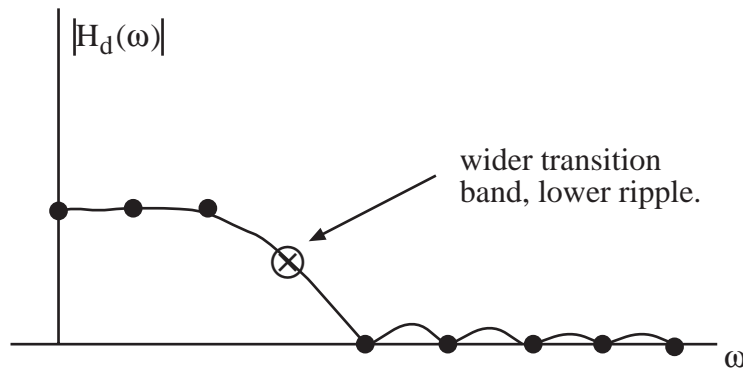
Note: If you take $G_d(\omega)$ to have zero phase, the designed H_d will still pass through the correct samples of G_d, but the response will be <u>terrible</u> off the grid $\omega = \dfrac{2\pi m}{N}$. Hoping for zero phase (no delay) in a causal filter is a pipe dream!

Example

Use the frequency sampling method to design a linear-phase HPF $\{h_n\}_{n=0}^{60}$, with cutoff $\omega_c = .6\pi$.

<u>Solution</u>

$$N = 61 \Rightarrow \text{phase} = -\,\frac{N-1}{2}\,\omega = -30\,\omega$$

$$\Rightarrow \quad G_d(\omega) = \begin{cases} 0 & 0 \le \omega < .6\pi \\ e^{-j30\omega} & .6\pi \le \omega \le 1.4\pi \\ 0 & 1.4\pi < \omega \le 2\pi \end{cases}$$

$$\Rightarrow \quad G_d\left(\frac{2\pi m}{61}\right) = \begin{cases} 0 & 0 \le m \le 18 \\ e^{-j30\frac{2\pi}{61}m} & 19 \le m \le 42 \\ 0 & 43 \le m \le 60 \end{cases}$$

Now, $\{h_n\}$ is the inverse DFT of $\left\{G_d\left(\dfrac{2\pi m}{61}\right)\right\}$ as given by

$$h_n = \frac{1}{61}\sum_{m=0}^{60} G_d\left(\frac{2\pi m}{61}\right) e^{j\frac{2\pi}{61}nm}$$

$$= \frac{1}{61} \sum_{m=19}^{42} e^{-j30\frac{2\pi}{61}m} \, e^{j\frac{2\pi}{61}nm}$$

$$= \frac{1}{61} \sum_{m=19}^{42} e^{j\frac{2\pi}{61}(n-30)m}$$

effect of linear phase is to circularly shift the $\{h_n\}$ since replacing
n–30 with $<n-30>_{61}$ does not change this equation

$$= \frac{1}{61} \sum_{k=0}^{23} e^{j\frac{2\pi}{N}(n-30)(k+19)}$$

$$k = m - 19$$

$$= \frac{1}{61} e^{j\frac{2\pi}{61}(n-30)19} \, \frac{1 - e^{j\frac{2\pi}{61}(n-30)24}}{1 - e^{j\frac{2\pi}{61}(n-30)}}$$

$$= \frac{\frac{1}{61} e^{j\frac{2\pi}{61}(n-30)31}}{e^{j\frac{2\pi}{61}(n-30)}} \cdot \frac{e^{-j\frac{2\pi}{61}(n-30)12} - e^{j\frac{2\pi}{61}(n-30)12}}{e^{-j\frac{\pi}{61}(n-30)} - e^{j\frac{\pi}{61}(n-30)}}$$

$$= \frac{1}{61} e^{j\pi(n-30)} \, \frac{-2j\sin\left(\frac{2\pi}{61}(n-30)12\right)}{-2j\sin\left(\frac{\pi}{61}(n-30)\right)}$$

$$\boxed{= \frac{1}{61}(-1)^n \, \frac{\sin\left(24\frac{\pi}{61}(n-30)\right)}{\sin\left(\frac{\pi}{61}(n-30)\right)} \qquad 0 \le n \le 60}$$

sampled periodic sinc

Here, the factor $(-1)^n$ makes this a high-pass filter rather than a low-pass filter. Since this term

multiplies a periodic sinc, which is similar to a sinc, we expect that the resulting frequency

response will not be too different from that obtained using the window design procedure with a truncation window.

Let's next consider the design of a low-pass filter $\{h_n\}_{n=0}^{N-1}$ where N is even. This example will be more complicated than the previous one in two respects. First, since N is even, the formula for $G_d(\omega)$ will take two different forms on $0 \leq \omega < 2\pi$. Second, since this is a low-pass filter, there will be two separate bands where $G_d(\omega)$ is nonzero on $0 \leq \omega < 2\pi$.

Example

Frequency sampling design of $\{h_n\}_{n=0}^{21}$, linear-phase LPF with $\omega_c = \dfrac{\pi}{2}$.

Solution

$$N = 22 \Rightarrow \text{phase} = -\frac{N-1}{2}\,\omega = \frac{-21}{2}\,\omega$$

$$\Rightarrow \quad G_d(\omega) = \begin{cases} e^{-j\frac{21}{2}\omega} & |\omega| \leq \dfrac{\pi}{2} \\ 0 & \dfrac{\pi}{2} < |\omega| \leq \pi \end{cases}$$

But, for a frequency sampling design, we need samples of $G_d(\omega)$ on $\omega \, \varepsilon \, [0, 2\pi)$. For N even, with $\dfrac{N-1}{2}$ noninteger, we must be careful! Consider the phase of $G_d(\omega)$. Here for clarity, we will not wrap the phase inside the interval $(-\pi, \pi)$ and will instead show its linear extension.

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$$G_d(\omega) = e^{-j\frac{21}{2}\omega} \text{ in here} \qquad G_d(\omega) = e^{-j\frac{21}{2}(\omega - 2\pi)}$$
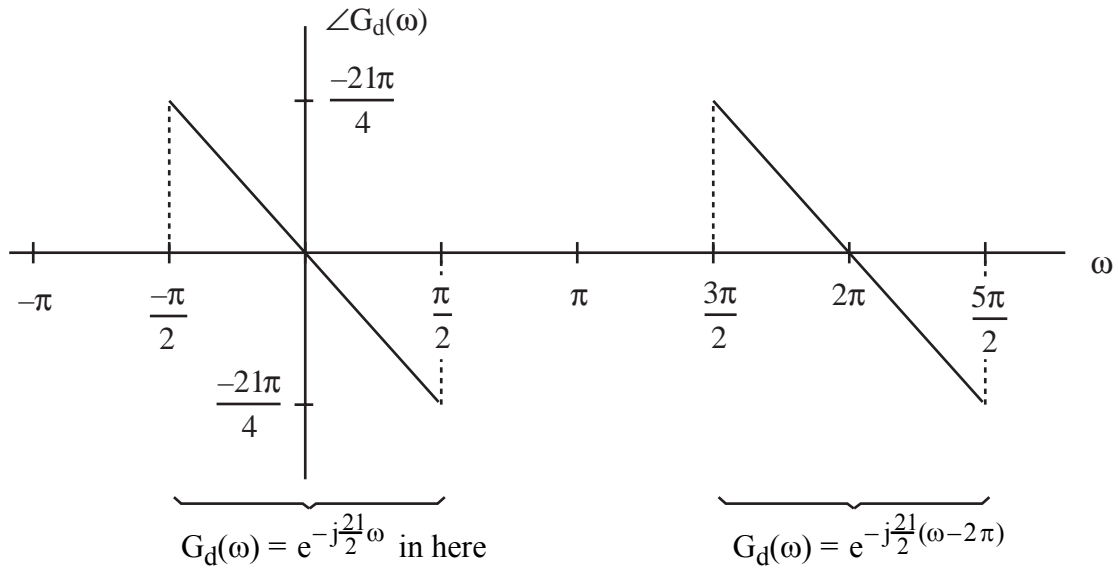
$$= e^{j\pi}\, e^{-j\frac{21}{2}\omega}$$

$$= -e^{-j\frac{21}{2}\omega}$$

in here
Note: Minus sign doesn't
occur if N is odd.

So:

$$G_d(\omega) \;=\; \begin{cases} e^{-j\frac{21}{2}\omega} & 0 \le \omega \le \dfrac{\pi}{2} \\[2mm] 0 & \dfrac{\pi}{2} < \omega < \dfrac{3\pi}{2} \\[2mm] -e^{-j\frac{21}{2}\omega} & \dfrac{3\pi}{2} \le \omega \le 2\pi \end{cases}$$

$$\Rightarrow G_d\left(\frac{2\pi m}{22}\right) \;=\; \begin{cases} e^{-j\frac{21}{2}\left(\frac{2\pi m}{22}\right)} & 0 \le m \le 5 \\[2mm] 0 & 6 \le m \le 16 \\[2mm] -e^{-j\frac{21}{2}\left(\frac{2\pi m}{22}\right)} & 17 \le m \le 21 \end{cases}$$

Thus,

$$h_n = \frac{1}{22} \sum_{m=0}^{5} e^{-j\frac{21}{2}\left(\frac{2\pi m}{22}\right)} e^{j\frac{2\pi}{22}mn} + \frac{1}{22} \sum_{m=17}^{21} -e^{-j\frac{21}{2}\left(\frac{2\pi m}{22}\right)} e^{j\frac{2\pi}{22}mn}$$

$$= \frac{1}{22} \sum_{m=0}^{5} e^{j\frac{2\pi}{22}\left(n-\frac{21}{2}\right)m} - \frac{1}{22} \sum_{m=17}^{21} e^{j\frac{2\pi}{22}\left(n-\frac{21}{2}\right)m}$$

Making the change of variable $\ell = m - 22$, the second sum can be rewritten as

$$\sum_{\ell=-5}^{-1} e^{j\frac{2\pi}{22}\left(n-\frac{21}{2}\right)(\ell+22)} = \sum_{\ell=-5}^{-1} e^{j\frac{2\pi}{22}\left(n-\frac{21}{2}\right)\ell} \underbrace{e^{-j21\pi}}_{=-1}$$

Thus,

$$h_n = \frac{1}{22} \sum_{m=-5}^{5} e^{j\frac{2\pi}{22}\left(n-\frac{21}{2}\right)m}$$

$$\underset{\substack{\uparrow \\ k = m + 5}}{=} \frac{1}{22} \sum_{k=0}^{10} e^{j\frac{2\pi}{22}\left(n-\frac{21}{2}\right)(k-5)}$$

$$= \frac{1}{22} e^{j\frac{2\pi}{22}\left(n-\frac{21}{2}\right)(-5)} \frac{1 - e^{j\frac{2\pi}{22}\left(n-\frac{21}{2}\right)11}}{1 - e^{j\frac{2\pi}{22}\left(n-\frac{21}{2}\right)}}$$

$$= \frac{1}{22} e^{-j\frac{10\pi}{22}\left(n-\frac{21}{2}\right)} \frac{e^{j\frac{2\pi}{22}\left(n-\frac{21}{2}\right)\frac{11}{2}}}{e^{j\frac{2\pi}{22}\left(n-\frac{21}{2}\right)\frac{1}{2}}} \bullet \frac{-j2\sin\frac{11\pi}{22}\left(n-\frac{21}{2}\right)}{-j2\sin\frac{\pi}{22}\left(n-\frac{21}{2}\right)}$$

$$\boxed{= \left| \frac{1}{22} \frac{\sin\frac{\pi}{2}\left(n-\frac{21}{2}\right)}{\sin\frac{\pi}{22}\left(n-\frac{21}{2}\right)} \right| \quad 0 \le n \le 21}$$

Thus, h_n is a sampled, periodic sinc, which is similar to the sinc-type design that would have resulted from a window design using a truncation window.

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In general, the formula for the coefficients of either an FIR LPF or HPF, designed using the frequency sampling technique, are given by

$$\text{LPF: } h_n = \frac{1}{N} \frac{\sin\left(\gamma \frac{\pi}{N}\left(n - \frac{N-1}{2}\right)\right)}{\sin\left(\frac{\pi}{N}\left(n - \frac{N-1}{2}\right)\right)}$$

$$\text{HPF: } h_n = (-1)^n \frac{1}{N} \frac{\sin\left(\gamma \frac{\pi}{N}\left(n - \frac{N-1}{2}\right)\right)}{\sin\left(\frac{\pi}{N}\left(n - \frac{N-1}{2}\right)\right)}$$

where γ is the number of samples $G_d\left(\frac{2\pi m}{N}\right)$, $0 \le m \le N - 1$, that are nonzero (number of samples in the passband).