Chapter 2

CT and DT Signal Representations

- 1 Fourier Series
- 2 Fourier Transform
- 3 Discrete Fourier Series
- 4 Discrete-Time Fourier Transform
- 5 Discrete Fourier Tranform
- 6 Applications

GOALS

1. Development of representations of CT and DT signals in the frequency domain.
2. Familiarization with Fourier Series and transform representations for CT and DT signals.

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Since the signals we encounter in engineering, science, and everyday life are as varied as the applications in which we engage them, it is often helpful to first study these applications in the presence of simplified versions of these signals. Much like a child learning to play an instrument for the first time, it is easier to start by attempting to play a single note before an entire musical score. Then, after learning many notes, the child becomes a musician and can synthesize a much broader class of music, building up from many notes. This approach of building-up our understanding of complex concepts by first understanding their basic building blocks is a fundamental precept of engineering and one that we will use frequently throughout this book.

In this chapter, we will explore signals in both continuous time and discrete time, together with a number of ways in which these signals can be built-up from simpler signals. Simplicity is in the eye of the beholder and what makes a signal appear simple in one context may not shed much light in another context. Many of the concepts we will develop throughout this text arise from studying large classes of signals, one building block at a time, and extrapolating system (or application) level behavior by considering the whole as a sum of its parts. In this chapter, we will focus specifically on sinusoidal signals as our basic building blocks as we consider both periodic and aperiodic signals in continuous and discrete time. Along this path, we will encounter the Fourier series representations of periodic signals as well as Fourier transform representations of aperiodic, infinite-length signals. In later chapters, we will find that so-called "time-domain" representations of signals sometimes prove more fruitful, and for discrete-time signals there is a natural way to construct signals one sample at a time.

2.1 Fourier Series representation of finite-length and periodic CT signals

In many applications in science and engineering, we often work with signals that are periodic in time. That is, the signal repeats itself over and over again with a given period of repetition. Examples of periodic signals might include the acoustic signal that emenates from a musical instrument, such as a trumpet when a single sustained note is played, or the vertical displacement of a mass in a frictionless spring-mass oscillator set into motion, or the horizontal displacement of a pendulum swaying to and fro in the absence of friction.

Mathematically, we represent a periodic signal, $x(t)$, as one whose value repeats at a fixed interval of time from the present. This interval, denoted T below, is called the "period" of the signal, and we express this relationship

$$x(t) = x(t+T), \text{ for all } t. \tag{2.1}$$

Equation (2.1) will, in general, be satisfied for a countably infinite number of possible values of T when $x(t)$ is periodic. The smallest, positive value of T for which Eq. (2.1) is satisfied, is called the "fundamental period" of the signal $x(t)$. For sinusoidal signals, such as

$$x(t) = \sin(\omega_0 t + \phi), \tag{2.2}$$

we can relate the frequency of oscillation, ω_0 to the fundamental period, T. This can be computed by noting that sinusoidal functions are equal when their arguments are either equal or differ only through a multiple of 2π, i.e.

$$\begin{aligned} x(t) &= x(t+T) \\ \sin(\omega_0 t + \phi) &= \sin(\omega_0(t+T) + \phi) \\ \sin(\omega_0 t + \phi + 2k\pi) &= \sin(\omega_0(t+T) + \phi) \\ \sin(\omega_0(t + 2k\pi/\omega_0) + \phi) &= \sin(\omega_0(t+T) + \phi) \end{aligned} \tag{2.3}$$

which, for $k = 1$, yields the relationship

$$T = 2\pi/\omega_0, \tag{2.4}$$

between the fundamental period, T, and the "fundamental frequency" ω_0. By analogy to sinusoidal signals, we refer to the value of $\omega_0 = 2\pi/T$ as the fundamental frequency of any signal that is periodic with a fundamental period T.

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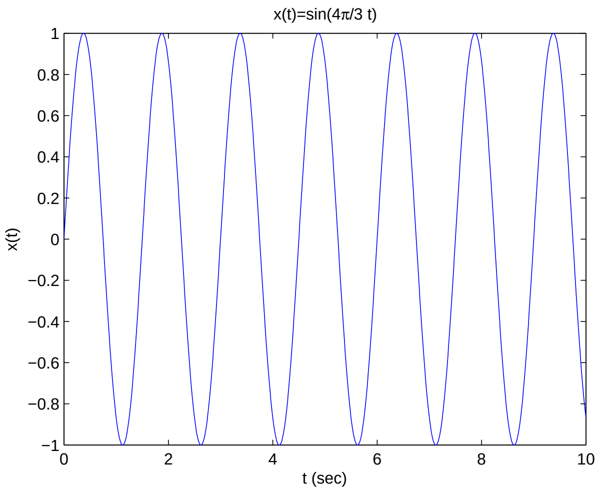

Figure 2.1: The periodic sinusoidal signal $x(t) = \sin((4\pi/3)t)$.

Here, we will provide a number of examples of periodic signals in continuous-time, including sinusoidal, square wave, traingular wave and complex exponential signals. By noting that any two periodic signals, $x(t)$ and $y(t)$ with the same period T can be added together to produce a new periodic signal of the same period, i.e.,

$$\begin{aligned} s(t) =& x(t) + y(t) \\ s(t+T) =& x(t+T) + y(t+T) = s(t+T), \end{aligned}$$

in 1807 Jean Baptiste Fourier (1807) considered the notion of building a large set of periodic signals from sinusoidal signals sharing the same period. Ignoring the phase,ϕ, for now, note that from (2.4), sinusoidal signals that share the same period must have fundamental frequencies given by $k\omega_0 = 2k\pi/T$ for different values of k. If two sinusoidal signals shared the same fundamental frequency, then they would be the same sinusoidal signal (recall that, for now, we are neglecting the phase, ϕ). We call such sinusoidal signals whose fundamental frequencies $k\omega_0$ are integer multiples of one fundamental frequency, harmonically-related sinusoids. Such harmonically-related sinusoids could indeed share the period, $2\pi/\omega_0$ while they would have different fundamental freqeuncies and hence different "fundamental periods."

We now consider how we might build-up a larger class of periodic signals from the basic building blocks of harmonically-related sinusoids. To extend our discussion to include complex-valued signals, we will employ Euler's relation to construct complex exponential signals of the form

$$\begin{aligned} x(t) =& e^{j(\omega_0 t + \phi)} \\ =& \cos(\omega_0 t + \phi) + j\sin(\omega_0 t + \phi) \end{aligned} \tag{2.5}$$

and in doing so, we can push the phase out of the picture so that it can be absorbed in a complex scalar constant out front, i.e.

$$x(t) = ce^{j\omega_0 t},$$

where, $c = e^{j\phi}$ is simply a complex constant whose effects on the sinusoidal nature of the signal have been conveniently parked outside the discussion. Complex-exponential signals of the form (2.5) are periodic with fundamental frequency $\omega_0 = 2\pi/T$ since they are simply constructed by pairing the real-valued periodic signal $\cos(\omega_0 t)$ with the purely imaginary signal $j\sin(\omega_0 t)$.

By simply adding together harmonically-related sinusoidal signals, we can construct a large class of periodic waveforms of amazing variety. For example, in Figure 2.2, note how by taking odd-valued harmonics (sinusoids with harmonically-related fundamental frequencies that are odd multiples of a single frequency, $\omega_0 = 2\pi$), we obtain an increasingly improving approximation to a square wave with unit period.

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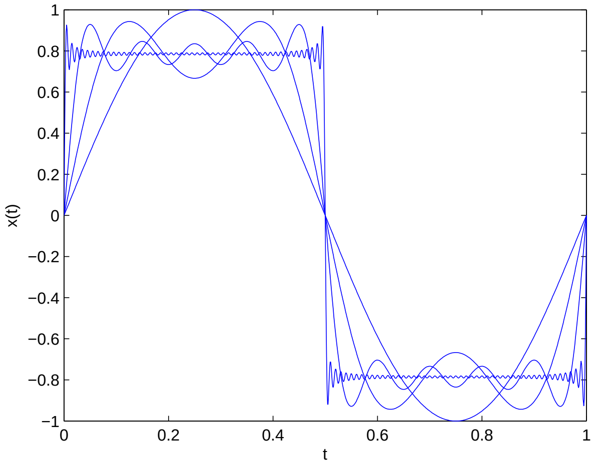

Figure 2.2: The periodic sinusoidal signal $x(t) = \sum_{k=1}^{N} \frac{1}{k}\sin(2k\pi t)$, for $k = 1, 3, 9$ and 99.

Generalizing this idea, we can explore the class of signals that can be constructed by such harmonically-related complex exponentials of the form

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}. \tag{2.6}$$

To bring the period of the periodic signal $x(t)$ into the equation, (2.6) is often written

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi kt/T}, \tag{2.7}$$

where $T = 2\pi/\omega_0$ is the fundamental period and ω_0 is the fundamental frequency of the periodic signal $x(t)$. The construction in (2.7) is referred to as the continuous-time Fourier series (CTFS) representation of $x(t)$ and (2.7) is often called the continuous-time Fourier series synthesis equation.

The Fourier series coefficients $X[k]$ can be obtained by multiplying (2.7) by $e^{-j2\pi kt/T}$ and integrating over a period of duration T to obtain

$$\int_0^T x(t)e^{-j2\pi kt/T}dt$$
$$= \int_0^T \left(\sum_{m=-\infty}^{\infty} X[m]e^{j2\pi(m-k)t/T} \right) dt,$$

where the limits of integration indicate that the we have chosen to evaluate the integral over the period $0 \leq t \leq T$. Note the use of the dummy variable m in the summation for the CTFS, since the variable k was already in use. To use k again would invite disaster into our derivation. Interchanging the order of integration and summation (which can be done under suitable conditions on the summation), we obtain,

$$\int_0^T x(t)e^{-j2\pi kt/T}dt$$
$$= \sum_{m=-\infty}^{\infty} \int_0^T X[m]e^{j2\pi(m-k)t/T}dt. \tag{2.8}$$

To proceed, we need to evaluate the integral

$$\begin{aligned}\int_0^T e^{j2\pi(m-k)t/T}dt =& \frac{T}{j2\pi(m-k)}e^{j2\pi(m-k)t/T}\Big|_0^T \\ =& \frac{T}{j2\pi(m-k)}[e^{j2\pi(m-k)}-1] \\ =& T\delta[m-k],\end{aligned}$$

where the second line arises from simple integration of an exponential function. The second line is readily seen to be equal to zero when $m \neq k$ and though one might be tempted to evaluate this line for $m = k$ (using a formula bearing the name of a famous 17th-century French mathematician), our efforts will be better spent setting $m = k$ into the integrand on the left hand side of the first line, from which we obtain

$$\int_0^T 1dt = T.$$

An interpretation of this result is that integration of a periodic complex exponential over an integer multiple, $(m-k)$, of its fundamental period, in this case $T/2\pi(m-k)$, is zero. The only periodic complex exponential that survives integration over the period T is the DC, i.e. $m = k$, term.

We can now return to (2.8) and apply this result, to obtain

$$\begin{aligned}\int_0^T x(t)e^{-j2\pi kt/T}dt =& \sum_{m=-\infty}^{\infty} X[m]T\delta[m-k] \\ =& TX[k],\end{aligned} \tag{2.9}$$

by the sifting property of the Kronocker delta function. We can now turn (2.9) around to obtain the continuous-time Fourier series analysis equation,

$$X[k] = \frac{1}{T}\int_0^T x(t)e^{-j2\pi kt/T}dt. \tag{2.10}$$

Putting the synthesis and analysis equations together, we have the continuous-time Fourier series representation of a periodic signal $x(t)$ as

CT Fourier Series Representation of a Periodic Signal

$$X[k] = \frac{1}{T}\int_0^T x(t)e^{-j\frac{2\pi kt}{T}}dt \tag{2.11}$$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi kt}{T}} \tag{2.12}$$

Example: CTFS of a Square Wave

Let us return to the square wave signal that we visited in Figure 2.2. In the figure, we appeared to have a method for constructing the periodic signal that, in the interval $[0, 1]$, satisfies

$$x(t) = \begin{cases} 1, & 0 \leq t \leq 0.5 \\ -1 & \text{else.} \end{cases} \tag{2.13}$$

Using (2.10), we obtain,

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$$\begin{aligned} X[k] &= \int_0^1 x(t)e^{-j2\pi kt}dt \\ &= \int_0^{0.5} e^{-j2\pi kt}dt - \int_{0.5}^1 e^{-j2\pi kt}dt \\ &= \frac{-1}{j2\pi k}\left([e^{-j\pi k}-1]-[1-e^{-j\pi k}]\right) \\ &= \frac{-1}{j2\pi k}2[(-1)^k-1] \\ &= \begin{cases} 0, & k \text{ even} \\ \frac{2}{j\pi k} & k \text{ odd.} \end{cases} \end{aligned} \tag{2.14}$$

Note that the $k = 0$ case can be readily evaluated by considering the integral in (2.14) for which the integral can be easily seen to vanish by the antisymmetry of $x(t)$ over the unit interval.

2.1.1 CT Fourier Series Properties

We have now been properly introduced to a method for building-up continuous-time periodic signals from a class of simple sinusoidal signals in (2.11)and a method for analysing the make-up of such periodic signals in terms of their constituent sinusoidal components in (2.12). Now that introductions are out of the way, we can explore some of the many useful properties of the CTFS representation. As we shall see, it is often helpful to consider the properties of a whole signal by virtue of the properties of its parts, and the relations we develop next will often prove useful in this process.

2.1.1.1 Linearity

The CTFS can be viewed as a linear operation, in the following manner. When two signals $x(t)$ and $y(t)$ are each constructed from their constituent sinusoidal signals according to the CTFS synthesis equation (2.12), the linear combination of these signals, $z(t) = ax(t) + by(t)$, for a, b real-valued constants, can be readily obtained by combining the constituent sinusoidal signals through the same linear combination. More specifically, when $x(t)$ is a periodic signal with CTFS coefficients $X[k]$ and $y(t)$ is a periodic signal with CTFS coefficients $Y[k]$ then the signal $z(t) = ax(t) + by(t)$ has CTFS coefficients given by $Z[k] = aX[k] + bY[k]$. The linearity property of the CTFS can be compactly represented as follows

$$x(t) \overset{CTFS}{\longleftrightarrow} X[k], y(t) \overset{CTFS}{\longleftrightarrow} Y[k] \implies z(t) = ax(t) + by(t) \overset{CTFS}{\longleftrightarrow} aX[k] + bY[k].$$

This result can be readily shown by substituting $z(t) = ax(t)+by(t)$ into the integral in (2.11) and expanding the integral into the two separate terms, one for $X[k]$ and one for $Y[k]$.

2.1.1.2 Time Shift

When a sinusoidal signal $x(t) = \sin(\omega_0 t)$ is shifted in time, the resulting signal $x(t - t_0)$ can be represented in terms of a simple phase shift of the origional sinusoidal signal, i.e. $x(t - t_0) = \sin(\omega_0(t - t_0)) = \sin(\omega_0 t - \phi)$,where $\phi = \omega_0 t_0 = 2\pi t_0/T$. Periodic signals that can be represented using the CTFS contain many, possibly infinitely many, sinusoidal (or complex exponential) signals. When such periodic signals are delayed in time, each of the constituent sinusoidal components of the signal are delayed by the same amount, however this translates into a different phase shift for each component. This can be readily seen from the CTFS analysis equation (2.11), as follows. For the signal $y(t) = x(t - t_0)$, we have

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$$
\begin{aligned}
Y[k] &= \frac{1}{T}\int_{t=0}^{T} x(t-t_0)e^{-j\frac{2\pi k}{T}t}dt \\
&= \frac{1}{T}\int_{s=-t_0}^{T-t_0} x(s)e^{-j\frac{2\pi k}{T}(s+t_0)}dt \\
&= \frac{1}{T}\int_{s=-t_0}^{0} x(s)e^{-j\frac{2\pi k}{T}(s+t_0)}ds + \frac{1}{T}\int_{s=0}^{T-t_0} x(s)e^{-j\frac{2\pi k}{T}(s+t_0)}ds \\
&= \frac{1}{T}\int_{s=-t_0}^{0} x(s+T)e^{-j\frac{2\pi k}{T}(s+T+t_0)}ds + \frac{1}{T}\int_{s=0}^{T-t_0} x(s)e^{-j\frac{2\pi k}{T}(s+t_0)}ds \\
&= \frac{1}{T}\int_{\tau=T-t_0}^{T} x(\tau)e^{-j\frac{2\pi k}{T}(\tau+t_0)}d\tau + \frac{1}{T}\int_{s=0}^{T-t_0} x(s)e^{-j\frac{2\pi k}{T}(s+t_0)}ds \\
&= \frac{1}{T}\int_{t=0}^{T} x(t)e^{-j\frac{2\pi k t_0}{T}}e^{-j\frac{2\pi k}{T}t}dt \\
&= X[k]e^{-j\frac{2\pi k t_0}{T}},
\end{aligned}
$$

where, the second line follows from the change of variable, $s = t - t_0$, the fourth line follows from the periodicity of both the signal $x(t)$ and the signal $e^{-j2\pi kt/T}$with period T, the fifth line follows from the change of variable $\tau = s + T$, and the last line follows from the definition of $X[k]$ after factoring the linear phase term $e^{-j2\pi kt_0/T}$ out of the integral. The time shift property of the CTFS can be compactly represented as follows

$$
x(t) \overset{CTFS}{\longleftrightarrow} X[k] \implies y(t) = x(t-t_0) \overset{CTFS}{\longleftrightarrow} X[k]e^{-j\frac{2\pi k}{T}t_0}.
$$

We see that a shift in time of a periodic signal corresponds to a modulation in frequency by a phase term that is linear with frequency with a slope that is proportional to the delay. This can be made easier if we adopt the convenient, but conceptually more challenging concept of integration over a period for the definition of the CTFS.

2.1.1.3 Frequency Shift

When a periodic signal $x(t)$ has a CTFS representation given by $X[k]$, a natural question that might arise is the what happens when the shifting that was discussed in section 2.1.1.2 is applied to the CTFS representation, $X[k]$. Specifically, if a periodic signal $y(t)$ were known to have a CTFS representation given by $Y[k] = X[k - k_0]$, it is interesting to understand the relationship in the time-domain between $y(t)$ and $x(t)$. This can be readily seen through examination of the CTFS analysis equation,

$$
\begin{aligned}
Y[k] &= X[k-k_0] \\
&= \frac{1}{T}\int_{t=0}^{T} x(t)e^{-j\frac{2\pi}{T}(k-k_0)t}dt \\
&= \frac{1}{T}\int_{t=0}^{T} x(t)e^{j\left(\frac{2\pi k_0}{T}\right)t}e^{-j\frac{2\pi}{T}kt}dt \\
&= \frac{1}{T}\int_{t=0}^{T} \left(x(t)e^{j\left(\frac{2\pi k_0}{T}\right)t}\right)e^{-j\frac{2\pi}{T}kt}dt,
\end{aligned}
$$

which leads to the relation

$$
x(t) \overset{CTFS}{\longleftrightarrow} X[k] \implies y(t) = x(t)e^{jk_0\omega_0 t} \overset{CTFS}{\longleftrightarrow} X[k-k_0],
$$

where $\omega_0 = \frac{2\pi}{T}$. We observe that a shift in the continuous time Fourier series coefficients by an integer amount k_0 corresponds to a modulation in the time domain signal $x(t)$ by a term whose frequency is proportional to the shift amount.

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2.1.1.4 Time Reversal

When a periodic signal $x(t) = e^{j2\pi t/T}$ is time-reversed, i.e. $y(t) = x(-t)$, the effect on its CTFS representation can be simply observed

$$\begin{aligned} X[k] &= \frac{1}{T}\int_{t=0}^{T} e^{j\frac{2\pi}{T}t}e^{-j\frac{2\pi k}{T}t}dt \\ &= \begin{cases} 1, & \text{for } k = 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} Y[k] &= \frac{1}{T}\int_{t=0}^{T} e^{-j\frac{2\pi}{T}t}e^{-j\frac{2\pi k}{T}t}dt \\ &= \begin{cases} 1, & \text{for } k = -1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

More generally, from the CTFS synthesis equation,

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi k}{T}t},$$

we see that by simply changing the sign of the time variable t, we obtain the general relation

$$\begin{aligned} y(t) &= x(-t) = \sum_{k=-\infty}^{\infty} X[k]e^{-j\frac{2\pi k}{T}t} \\ &= \sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi(-k)}{T}t} \\ &= \sum_{m=-\infty}^{\infty} X[-m]e^{j\frac{2\pi k}{T}t}, \end{aligned}$$

yielding the relation

$$x(t) \overset{CTFS}{\longleftrightarrow} X[k] \implies y(t) = x(-t) \overset{CTFS}{\longleftrightarrow} X[-k],$$

i.e., changing the sign of the time axis corresponds to changing the sign of the CTFS frequency index.

2.1.1.5 Time Scaling

When a periodic signal undergoes a time-scale change, such as one that compresses the time axes, $y(t) = x(at)$, where $a > 1$ is a real-valued constant, the resulting signal $y(t)$ would remain periodic, however the period would change correspondingly, such that $y(t+T_y) = y(t)$ would be satisfied for a different period T_y. This can be easily seen by substituting in for $x(t)$ in the relation, $y(t) = x(at) = y(t+T_y) = x(a(t+T_y))$ and the noting that $x(at) = x(at+T)$, due to the periodicity of $x(t)$ with period T. This leads to the relation $x(a(t+T_y)) = x(at+T)$ or $T_y = T/a$. This makes intuitive sense, since the time-axis in the signal $y(t)$ has been compressed by a factor of a, therefore the time at which it will repeat must also have compressed by the same factor. Now, even though the period of the signal $y(t)$ has changed, we also are interested in the full CTFS representaiton of $y(t)$. This is given by

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$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi k}{T}at} \\ &= \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi k}{T_y}t}, \end{aligned}$$

where the second line follows from the definition of T_y. Note that although we have that

$$x(t) \overset{CTFS}{\longleftrightarrow} X[k] \implies y(t) = x(at) \overset{CTFS}{\longleftrightarrow} X[k],$$

that is the sequence of CTFS coefficients $Y[k]$ is identical to $X[k]$, the CTFS representation for $x(t)$ and $y(t)$ differ substantially, since they are defined for completely different periods, $T \neq T_y$. As a result, the fundamental frequency for the periodic signal $x(t)$ is $2\pi/T$, which is different from that of $y(t)$, which is $2\pi a/T$. Hence, the frequency content of the signals differ substantially.

2.1.1.6 Conjugate Symmetry

The effect of conjugating a complex-valued signal on its CTFS representation can be seen by simply conjugating the CTFS synthesis relation,

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi k}{T}t} \\ x^*(t) &= \left(\sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi k}{T}t} \right)^* \\ &= \sum_{k=-\infty}^{\infty} X^*[k] e^{-j\frac{2\pi k}{T}t} \\ &= \sum_{k=-\infty}^{\infty} X^*[k] e^{j\frac{2\pi(-k)}{T}t} \\ &= \sum_{m=-\infty}^{\infty} X^*[-m] e^{j\frac{2\pi m}{T}t} \end{aligned}$$

yielding that

$$x(t) \overset{CTFS}{\longleftrightarrow} X[k] \implies x^*(t) \overset{CTFS}{\longleftrightarrow} X^*[-k].$$

When the periodic signal $x(t)$ is real valued, i.e. $x(t)$ only takes on values that are real numbers, then the CTFS exhibits a symmetry property. This arises directly from the definintion of the CTFS, and that real numbers equal their conjugates, i.e. $x(t) = x^*(t)$, such that

$$x(t) = x^*(t) \overset{CTFS}{\longleftrightarrow} X[k] \implies X[k] = X^*[-k].$$

Note that when the signal is real-valued and is an even function of time, such that $x(t) = x(-t)$, then its CTFS is also real-valued and even, i.e. $X[k] = X^*[k] = X[-k]$. It can be shown by similar reasoning that when the signal is periodic, real-valued, and an odd function of time, that the CTFS coefficients are purely imaginary and odd, i.e. $X[k] = -X^*[k] = -X[-k]$.

2.1.1.7 Products of Signals

When two periodic signals of the same period are multiplied in time, such that $z(t) = x(t)y(t)$, the resulting signal remains periodic with the same period, such that $z(t) = x(t)y(t) = x(t+T)y(t+T) = z(t+T)$. Hence,

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each of the three signals admit CTFS representations using the same set of harmonically related signals. We can observe the effect on the resulting CTFS representation through the analysis equation,

$$\begin{aligned} Z[k] &= \frac{1}{T}\int_{t=0}^{T}(x(t)y(t))e^{-j\frac{2\pi k}{T}t}dt \\ &= \frac{1}{T}\int_{t=0}^{T}(y(t)\left(\sum_{m=-\infty}^{\infty}X[m]e^{j\frac{2\pi m}{T}t}\right))e^{-j\frac{2\pi k}{T}t}dt \\ &= \sum_{m=-\infty}^{\infty}X[m]\frac{1}{T}\int_{t=0}^{T}y(t)e^{-j\frac{2\pi(k-m)}{T}t}dt \\ &= \sum_{m=-\infty}^{\infty}X[m]Y[k-m]. \end{aligned}$$

The relationship between the CTFS coefficients for $z(t)$ and those of $x(t)$ and $y(t)$ is called a discrete convolution between the two sequences $X[k]$ and $Y[k]$,

$$x(t) \stackrel{CTFS}{\longleftrightarrow} X[k], y(t) \stackrel{CTFS}{\longleftrightarrow} Y[k] \implies z(t) = x(t)y(t) \stackrel{CTFS}{\longleftrightarrow} \sum_{m=-\infty}^{\infty} X[m]Y[k-m].$$

2.1.1.8 Convolution

A dual relationship to that of multiplication in time, is multiplication of CTFS coefficients. Specifically, when the two signals $x(t)$ and $y(t)$ are each periodic with period T, the periodic signal $z(t)$ of period T, whose CTFS representation is given by $Z[k] = X[k]Y[k]$ corresponds to a periodic convolution of the signals $x(t)$ and $y(t)$. This can be seen as follows,

$$\begin{aligned} z(t) &= \sum_{k=-\infty}^{\infty}(X[k]Y[k])\, e^{j\frac{2\pi k}{T}t} \\ &= \sum_{k=-\infty}^{\infty}\left(\frac{1}{T}\int_{\tau=0}^{T}x(\tau)e^{-j\frac{2\pi k}{T}\tau}d\tau\right)Y[k]e^{j\frac{2\pi k}{T}t} \\ &= \frac{1}{T}\int_{\tau=0}^{T}x(\tau)\left(\sum_{k=-\infty}^{\infty}Y[k]e^{j\frac{2\pi k}{T}(t-\tau)}\right)d\tau \\ &= \frac{1}{T}\int_{\tau=0}^{T}x(\tau)y(t-\tau)d\tau \end{aligned}$$

where the integral relationship in the last line is called periodic convolultion. This leads to the following property of the CTFS,

$$x(t) \stackrel{CTFS}{\longleftrightarrow} X[k], y(t) \stackrel{CTFS}{\longleftrightarrow} Y[k] \implies z(t) = \frac{1}{T}\int_{\tau=0}^{T} x(\tau)y(t-\tau)d\tau \stackrel{CTFS}{\longleftrightarrow} Z[k] = X[k]Y[k].$$

2.1.1.9 Integration

When the signal $y(t)$ and $x(t)$ are related through a running integral, i.e. $y(t) = \int_{\tau=0}^{t} x(\tau)d\tau$, we can relate their CTFS as follows,

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$$
\begin{aligned}
x(t) = \frac{d}{dt}\int_{\tau=0}^{t} x(\tau)d\tau &= \frac{d}{dt}\int_{\tau=0}^{t}\Big(\sum_{k=-\infty,k\neq 0}^{\infty} X[k]e^{j\frac{2\pi k}{T}\tau} + X[0]\Big)d\tau \\
&= \frac{d}{dt}\Big(\sum_{k=-\infty,k\neq 0}^{\infty} X[k]\int_{\tau=0}^{t} e^{j\frac{2\pi k}{T}\tau}d\tau + X[0]t\Big) \\
&= \frac{d}{dt}\sum_{k=-\infty,k\neq 0}^{\infty} X[k]\left[\frac{T}{j2\pi k}e^{j\frac{2\pi k}{T}\tau}\right]_{\tau=0}^{t} + X[0] \\
&= \frac{d}{dt}\sum_{k=-\infty,k\neq 0}^{\infty} X[k]\left[\frac{T}{j2\pi k}\left(e^{j\frac{2\pi k}{T}t} - 1\right)\right] + X[0] \\
&= \frac{d}{dt}\sum_{k=-\infty,k\neq 0}^{\infty} X[k]\frac{T}{j2\pi k}e^{j\frac{2\pi k}{T}t} - \frac{d}{dt}\sum_{k=-\infty,k\neq 0}^{\infty} X[k]\frac{T}{j2\pi k} + X[0] \\
&= \frac{d}{dt}\sum_{k=-\infty,k\neq 0}^{\infty}\left(X[k]\frac{T}{j2\pi k}\right)e^{j\frac{2\pi k}{T}t} + X[0] \\
&= \frac{d}{dt}\Big(\sum_{k=-\infty,k\neq 0}^{\infty} Y[k]e^{j\frac{2\pi k}{T}t} + X[0]t\Big) \\
\text{From this, if we let } y(t) &= \sum_{k=-\infty,k\neq 0}^{\infty}\left(X[k]\frac{T}{j2\pi k}\right)e^{j\frac{2\pi k}{T}t} + X[0]t \\
\frac{d}{dt}y(t) &= \sum_{k=-\infty,k\neq 0}^{\infty} X[k]e^{j\frac{2\pi k}{T}t} + X[0] \\
&= x(t).
\end{aligned}
$$

This yields the property,

$$
x(t) \overset{CTFS}{\longleftrightarrow} X[k] \implies y(t) = \int_{\tau=0}^{t} x(\tau)d\tau \overset{CTFS}{\longleftrightarrow} \begin{cases} \frac{T}{j2\pi k}X[k] & k \neq 0 \\ 0 & k = 0 \end{cases},
$$

where we must only consider $x(t)$ such that $X[0] = 0$, or else $y(t)$ would not be periodic.

2.1.1.10 Differentiation

Similarly, we can consider the relationship between $y(t) = \frac{d}{dt}x(t)$ and their corresponding CTFT representations. From the definition of the CTFS, we have

$$
\begin{aligned}
y(t) &= \frac{d}{dt}x(t) \\
&= \frac{d}{dt}\sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi k}{T}t} \\
&= \sum_{k=-\infty}^{\infty} X[k]\frac{d}{dt}e^{j\frac{2\pi k}{T}t} \\
&= \sum_{k=-\infty}^{\infty}\left(X[k]\frac{j2\pi k}{T}\right)e^{j\frac{2\pi k}{T}t}
\end{aligned}
$$

from which we obtain the relation

$$x(t) \overset{CTFS}{\longleftrightarrow} X[k] \implies y(t) = \frac{d}{dt}x(t) \overset{CTFS}{\longleftrightarrow} \left(\frac{j2\pi k}{T}\right) X[k].$$

2.1.1.11 Parseval's relation

The energy containted within a period of a periodic signal can also be computed in terms of its CTFS representation using Parseval's relation,

$$x(t) \overset{CTFS}{\longleftrightarrow} X[k] \implies \frac{1}{T}\int_{t=0}^{T} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X[k]|^2.$$

This relation can be derived using the definition of the CTFS as follows,

$$\begin{aligned}
\frac{1}{T}\int_{t=0}^{T} |x(t)|^2 dt &= \frac{1}{T}\int_{t=0}^{T} x(t)x^*(t)dt \\
&= \frac{1}{T}\int_{t=0}^{T} x(t)\left(\sum_{k=-\infty}^{\infty} X^*[-k]e^{j\frac{2\pi k}{T}t}\right) dt \\
&= \sum_{k=-\infty}^{\infty} X^*[-k]\left(\frac{1}{T}\int_{t=0}^{T} x(t)e^{j\frac{2\pi k}{T}t}dt\right) \\
&= \sum_{k=-\infty}^{\infty} X^*[-k]\left(\frac{1}{T}\int_{t=0}^{T} x(t)e^{-j\frac{2\pi(-k)}{T}t}dt\right) \\
&= \sum_{k=-\infty}^{\infty} X^*[-k]X[-k] \\
&= \sum_{m=-\infty}^{\infty} |X[m]|^2.
\end{aligned}$$

Parseval's relation shows that the energy in a period of a periodic signal is equal to the sum of the energies contained within each of the harmonic components that make up the signal through the CTFS representation.

2.2 Fourier transform representation of CT signals

Now that we have seen how we may build-up a large class of continuous-time periodic signals from the set of simpler complex exponential periodic signals, we return to apply this line of thinking to the more general class of continuous-time aperiodic (not periodic) signals. Just as was the case for periodic signals, a remakably rich class of aperiodic signals can also be constructed from linear combinations of complex exponentials. In the case of periodic continuous-time signals, since the signals of interest were periodic, the CTFS was restricted to contruct such signals through combinations of harmonically related exponentials. However for more general aperiodic signals, we may consider building an even larger class of signals by removing this restriction on the ingredients used to makeup a given signal. Since harmonically related complex exponentials can be enumerated, the CTFS took the form of a summation over the countably infinite set of all harmoically related exponentials of a given fundamental frequency. However, removing the restriction to only using harmonically related terms, the class of all possible complex exponentials arises from a continuum of possible freqeunecy components and the form used with which to contruct linear combinations will take the form of an integral, rather than an infinite summation. Just as with the continuous-time Fourier series, where the CTFS analysis equation provided a method for calculating the frequency components that makeup a given periodic signal, the continuous-time Fourier transform provides a method for calculating the spectrum of frequency components that makup an aperiodic signal from this class. The resulting integral used to contruct this large class of signals using this specific spectrum of frequency components is called the Fourier integral, or the continuous-time Fourier synthesis equation.

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One method for introducing the continous-time Fourier transform is through the CTFS. By considering continuous-time aperiodic signals as the result of taking continous-time periodic signals to the limit of an infinite period, we may observe how the CTFS transitions from a countable sum of harmonically-related complex exponentials, into a continuous integral over the continuum of possible frequencies. Let us return to the square wave signal that we visited in Figure 2.2. In this case, however, we will alter the signal to take the form

$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0 & \text{else} \end{cases}$$

over the unit interval, $t \in [0, 1]$. Using (2.10), we once again obtain its CTFS representation, however this time, we consider the period of repetition of the "on" period of the square wave to be given by the variable T, i.e. we have

$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0 & \text{else} \end{cases}$$

for $t \in [0, T]$, and then repeating every T seconds. This yields the following CTFS representation

$$\begin{aligned} X[k] &= \int_0^T x(t) e^{-j\frac{2\pi k}{T}t} dt \\ &= \int_0^1 e^{-j\frac{2\pi k}{T}t} dt \\ &= \frac{-T}{j2\pi k}\left(e^{-j\frac{2\pi k}{T}} - 1\right) \\ &= \frac{-T}{j2\pi k} e^{-j\frac{\pi k}{T}}\left(e^{-j\frac{\pi k}{T}} - e^{j\frac{\pi k}{T}}\right) \\ &= \frac{T}{j2\pi k} e^{-j\frac{\pi k}{T}} 2j\sin\left(\frac{\pi k}{T}\right) \\ &= \begin{cases} \frac{\sin\left(\frac{\pi k}{T}\right)}{\frac{\pi k}{T}} e^{-j\frac{\pi k}{T}} & k \neq 0 \\ 1 & k = 0, \end{cases} \end{aligned} \tag{2.15}$$

where the $k = 0$ term is once again determined by closer examination of the first line of the derivation, rather than attempting further analysis on the expression at containing vanishing terms in the numerator and demoninator. We consider the expression in (2.15) for various values of T in Figure 2.3. By plotting the magnitude of the CTFS coefficients $|X[k]|$ versus the harmonicaly related frequency components $\frac{2\pi k}{T}$ for various values of T, ranging from $T = 4$, up to $T = 32$, we see that the envelope containing the CTFS coefficients remains constant, while the CTFS coefficients move closer and closer to one another in absolute frequency.

The envelope that is observed in the figure, can be viewed as the value that the CTFS representation would take on as the period of the signal is made larger and larger. Recognizing this process, Fourier defined this envelope as

$$X(\omega) = \int_{t=-\infty}^{\infty} x(t) e^{-j\omega t} dt, \tag{2.16}$$

where the frequency variable ω takes on all values on the real line, and for which (2.16) is known as the continuous-time Fourier transform (CTFT). For this example, the continuous-time Fourier transform would evaluate to

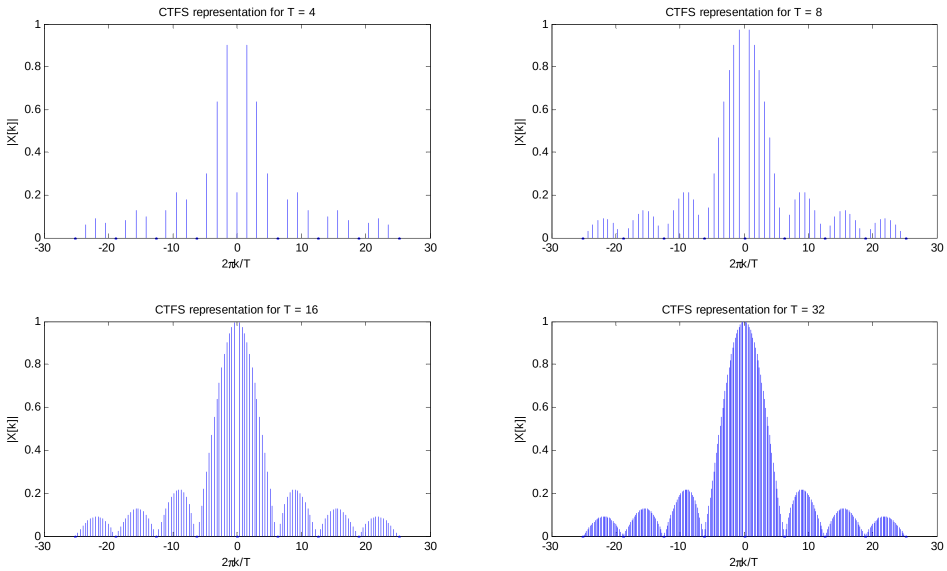

Figure 2.3: CTFS representation of the periodic signal in 2.17for $T = 4, 8, 16, 32$.

$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{0}^{1} e^{-j\omega t}dt \\
&= \frac{-1}{j\omega}\left(e^{-j\omega} - 1\right) \\
&= \frac{-1}{j\omega}e^{-j\omega/2}\left(e^{-j\omega/2} - e^{j\omega/2}\right) \\
&= \frac{1}{j\omega}e^{-j\omega/2}2j\sin\left(\omega/2\right) \\
&= \begin{cases} \frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}e^{-j\frac{\omega}{2}} & \omega \neq 0 \\ 1 & \omega = 0. \end{cases}
\end{aligned} \tag{2.17}
$$

While the CTFT analysis equation (2.16) provides the composition of any of a large class of signals through a linear superposition of complex exponential signals of the form $e^{j\omega t}$, the CTFT synthesis equation provides the recipe for constructing such signals from their constituent set, as

$$x(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega.$$

Together, the two expressions make up the CTFT representation for aperiodic signals,

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \\
X(\omega) &= \int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt
\end{aligned}
$$

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CT Fourier Transform Representation of Aperiodic Signals

$$x(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \quad (2.18)$$

$$X(\omega) = \int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt \quad (2.19)$$

2.2.1 CT Fourier Transform Properties

We have now been properly introduced to a method for building-up continuous-time aperiodic signals from a class of complex exponential signals in (2.18) and a method for analysing the make-up of such periodic signals in terms of their constituent sinusoidal components in (2.19). Once again, now that introductions are out of the way, we can explore some of the many useful properties of the CTFT representation. Many of the properties of the CTFT follow directly, or along similar lines, of those of the CTFS.

2.2.1.1 Linearity

The CTFT can be viewed as a linear operation, in the following manner. When two signals $x(t)$ and $y(t)$ are each constructed from their constituent complex exponential signals according to the CTFT synthesis equation, the linear combination of these signals, $z(t) = ax(t) + by(t)$, for a, b real-valued constants, can be readily obtained by combining the constituent complex exponential signals through the same linear combination. More specifically, when $x(t)$ is an aperiodic signal with CTFT coefficients $X(\omega)$ and $y(t)$ is an aperiodic signal with CTFT $Y(\omega)$ then the signal $z(t) = ax(t) + by(t)$ has a CTFT representation given by $Z(\omega) = aX(\omega) + bY(\omega)$. The linearity property of the CTFT can be compactly represented as follows

$$x(t) \stackrel{CTFT}{\longleftrightarrow} X(\omega), y(t) \stackrel{CTFT}{\longleftrightarrow} Y(\omega) \implies z(t) = ax(t) + by(t) \stackrel{CTFT}{\longleftrightarrow} aX(\omega) + bY(\omega).$$

2.2.1.2 Time Shift

For the signal $y(t) = x(t - t_0)$, we have

$$\begin{aligned} Y(\omega) &= \int_{t=-\infty}^{\infty} x(t - t_0)e^{-j\omega t}dt \\ &= \int_{s=-\infty}^{\infty} x(s)e^{-j\omega(s+t_0)}ds \\ &= \int_{s=-\infty}^{\infty} x(s)e^{-j\omega t_0}e^{-j\omega s}ds \\ &= X(\omega)e^{-j\omega t_0}, \end{aligned}$$

where, the second line follows from the change of variable, $s = t - t_0$. The time shift property of the CTFT can be compactly represented as follows

$$x(t) \stackrel{CTFT}{\longleftrightarrow} X(\omega) \implies y(t) = x(t - t_0) \stackrel{CTFT}{\longleftrightarrow} X(\omega)e^{-j\omega t_0}.$$

We see that a shift in time of an aperiodic signal corresponds to a modulation in frequency by a phase term that is linear with frequency with a slope that is proportional to the delay.

2.2.1.3 Frequency Shift

When a signal $x(t)$ has a CTFT representation given by $X(\omega)$, a natural question that might arise is the what happens when the shifting that was discussed in section 2.2.1.2 is applied to the CTFT representation, $X(\omega)$. Specifically, if a signal $y(t)$ were known to have a CTFT representation given by $Y(\omega) = X(\omega - \omega_0)$, it

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is interesting to understand the relationship in the time-domain between $y(t)$ and $x(t)$. This can be readily seen through examination of the CTFT analysis equation,

$$\begin{aligned} Y(\omega) &= X(\omega-\omega_0) \\ &= \int_{t=-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt \\ &= \int_{t=-\infty}^{\infty} \left(x(t)e^{j\omega_0 t}\right) e^{-j\omega t}dt, \end{aligned}$$

which leads to the relation

$$x(t) \overset{CTFT}{\longleftrightarrow} X(\omega) \implies y(t) = x(t)e^{j\omega_0 t} \overset{CTFT}{\longleftrightarrow} X(\omega-\omega_0).$$

We observe that a shift in the frequency of the continuous time Fourier transform by an amount ω_0 corresponds to a modulation in the time domain signal $x(t)$ by a term whose frequency is proportional to the shift amount.

2.2.1.4 Time Reversal

Analogous to the result for the CTFS, we have from the CTFT synthesis equation,

$$x(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega,$$

we see that by simply changing the sign of the time variable t, we obtain the general relation

$$\begin{aligned} y(t) &= x(-t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{-j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j(-\omega)t}d\omega \\ &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(-\omega)e^{j\omega t}d\omega, \end{aligned}$$

yielding the relation

$$x(t) \overset{CTFT}{\longleftrightarrow} X(\omega) \implies y(t) = x(-t) \overset{CTFT}{\longleftrightarrow} X(-\omega),$$

i.e., changing the sign of the time axis corresponds to changing the sign of the CTFT frequency index.

2.2.1.5 Time Scaling

When signal undergoes a time-scale change, such as one that compresses the time axes, $y(t) = x(at)$, where $a > 1$ is a real-valued constant, the resulting signal $y(t)$ is given by

$$\begin{aligned} y(t) &= \int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega at}d\omega \\ &= \int_{\nu=-\infty}^{\infty} \frac{1}{|a|}X(\nu/a)e^{j\nu t}d\nu, \end{aligned}$$

where the second line follows from the substitution $\nu = a\omega$. This yields the following relation for $y(t) = x(at)$,

$$x(t) \overset{CTFT}{\longleftrightarrow} X(\omega) \implies y(t) = x(at) \overset{CTFT}{\longleftrightarrow} \frac{1}{|a|}X(\omega/a).$$

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2.2.1.6 Conjugate Symmetry

The effect of conjugating a complex-valued signal on its CTFT representation can be seen by simply conjugating the CTFT synthesis relation,

$$\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \\
x^*(t) &= \left(\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega\right)^* \\
&= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X^*(\omega)e^{-j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X^*(\omega)e^{j(-\omega)t}d\omega \\
&= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X^*(-\omega)e^{j\omega t}d\omega
\end{aligned}$$

yielding that

$$x(t) \overset{CTFT}{\longleftrightarrow} X(\omega) \implies x^*(t) \overset{CTFT}{\longleftrightarrow} X^*(-\omega).$$

When the signal $x(t)$ is real valued, then the CTFT exhibits a symmetry property. This arises directly from the definintion of the CTFT, and that real numbers equal their conjugates, i.e. $x(t) = x^*(t)$, such that

$$x(t) = x^*(t) \overset{CTFT}{\longleftrightarrow} X(\omega) \implies X(\omega) = X^*(-\omega).$$

Note that when the signal is real-valued and is an even function of time, such that $x(t) = x(-t)$, then its CTFT is also real-valued and even, i.e. $X(\omega) = X^*(\omega) = X(-\omega)$. It can be shown by similar reasoning that when the signal real-valued, and an odd function of time, that the CTFT is purely imaginary and odd, i.e. $X(\omega) = -X^*(\omega) = -X(-\omega)$.

2.2.1.7 Products of Signals

When signals are multiplied in time, such that $z(t) = x(t)y(t)$, the resulting signal has a CTFS representation that can be obtained through the analysis equation,

$$\begin{aligned}
Z(\omega) &= \int_{t=-\infty}^{\infty} (x(t)y(t))e^{-j\omega t}dt \\
&= \int_{t=-\infty}^{\infty} (y(t)\left(\frac{1}{2\pi}\int_{\nu=-\infty}^{\infty} X(\nu)e^{j\nu t}d\nu\right))e^{-j\omega t}dt \\
&= \frac{1}{2\pi}\int_{\nu=-\infty}^{\infty} X(\nu)\left(\int_{t=-\infty}^{\infty} y(t)e^{-j(\omega-\nu)t}dt\right)d\nu \\
&= \frac{1}{2\pi}\int_{\nu=-\infty}^{\infty} X(\nu)Y(\omega-\nu)d\nu.
\end{aligned}$$

The relationship between the CTFT representation for $z(t)$ and those of $x(t)$ and $y(t)$ is seen to be a convolution between the two CTFTs $X(\omega)$ and $Y(\omega)$,

$$x(t) \overset{CTFT}{\longleftrightarrow} X(\omega), y(t) \overset{CTFT}{\longleftrightarrow} Y(\omega) \implies z(t) = x(t)y(t) \overset{CTFT}{\longleftrightarrow} \frac{1}{2\pi}\int_{\nu=-\infty}^{\infty} X(\nu)Y(\omega-\nu)d\nu.$$

2.2.1.8 Convolution

A dual relationship to that of multiplication in time, is multiplication of CTFT representations. Specifically, the signal whose CTFT representation is given by $Z(\omega) = X(\omega)Y(\omega)$ corresponds to a convolution of the signals $x(t)$ and $y(t)$. This can be seen as follows,

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$$\begin{aligned}
z(t) &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \left(X(\omega)Y(\omega)\right) e^{j\omega t} d\omega \\
&= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \left(\int_{\tau=-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau\right) Y(\omega)e^{j\omega t}d\omega \\
&= \int_{\tau=-\infty}^{\infty} x(\tau)\left(\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} Y(\omega)e^{j\omega(t-\tau)}\right) d\tau \\
&= \int_{\tau=-\infty}^{\infty} x(\tau)y(t-\tau)d\tau
\end{aligned}$$

where the integral relationship in the last line is recognized as a convolultion. This leads to the following property of the CTFT,

$$x(t) \overset{CTFT}{\longleftrightarrow} X(\omega), y(t) \overset{CTFT}{\longleftrightarrow} Y(\omega) \implies z(t) = \int_{\tau=-\infty}^{\infty} x(\tau)y(t-\tau)d\tau \overset{CTFT}{\longleftrightarrow} Z(\omega) = X(\omega)Y(\omega).$$

2.2.1.9 Integration

When the signal $y(t)$ and $x(t)$ are related through a running integral, i.e. $y(t) = \int_{\tau=-\infty}^{t} x(\tau)d\tau$, we can relate their CTFTs as follows,

$$x(t) \overset{CTFT}{\longleftrightarrow} X(\omega) \implies y(t) = \int_{\tau=-\infty}^{t} x(\tau)d\tau \overset{CTFT}{\longleftrightarrow} \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega),$$

where the relation is easiest shown using the differentiation property derived next together with the following observation. When $\omega = 0$, $Y(\omega)$ is unbounded if $X(0)$ is nonzero.

2.2.1.10 Differentiation

Similarly, we can consider the relationship between $y(t) = \frac{d}{dt}x(t)$ and their corresponding CTFT representations. From the definition of the CTFT, we have

$$\begin{aligned}
y(t) &= \frac{d}{dt}x(t) \\
&= \frac{d}{dt}\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}dt \\
&= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)\frac{d}{dt}e^{j\omega t}dt \\
&= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \left(j\omega X(\omega)\right) e^{j\omega t}dt
\end{aligned}$$

from which we obtain the relation

$$x(t) \overset{CTFT}{\longleftrightarrow} X(\omega) \implies y(t) = \frac{d}{dt}x(t) \overset{CTFT}{\longleftrightarrow} j\omega X(\omega).$$

2.2.1.11 Parseval's relation

The energy containted in a finite-energy signal (note that the CTFT exists in the case of finite energy signals, i.e. signals that can be square integrated) can also be computed in terms of its CTFT representation using Parseval's relation,

$$x(t) \overset{CTFT}{\longleftrightarrow} X(\omega) \implies \int_{t=-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} |X(\omega)|^2 d\omega.$$

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Section	CTFT Property	Continuous Time Signal	Continuous Time Fourier Transform
	Definition	$x(t)$	$X(\omega) = \int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt$
2.2.1.1	Linearity	$z(t) = ax(t) + by(t)$	$Z(\omega) = aX(\omega) + bY(\omega)$
2.2.1.2	Time Shift	$y(t) = x(t-T)$	$Y(\omega) = X(\omega)e^{-j\omega T}$
2.2.1.3	Modulation	$y(t) = x(t)e^{j\omega_0 t}$	$Y(\omega) = X(\omega - \omega_0)$
2.2.1.4	Time Reversal	$y(t) = x(-t)$	$Y(\omega) = X(-\omega)$
2.2.1.5	Time Scaling	$y(t) = x(at)$	$Y(\omega) = \frac{1}{\lvert a\rvert}X(\omega/a)$
2.2.1.6	Conjugate Symmetry	$x(t) = x^*(t)$	$X(\omega) = X^*(-\omega)$
2.2.1.7	Products of Signals	$z(t) = x(t)y(t)$	$Z(\omega) = \frac{1}{2\pi}\int_{\nu=-\infty}^{\infty} X(\nu)Y(\omega-\nu)d\nu$
2.2.1.8	Convolution	$z(t) = \int_{\tau=-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$	$Z(\omega) = X(\omega)Y(\omega)$
2.2.1.9	Integration	$y(t) = \int_{\tau=-\infty}^{t} x(\tau)d\tau$	$Y(\omega) = \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$.
2.2.1.10	Differentiation	$y(t) = \frac{d}{dt}x(t)$	$Y(\omega) = j\omega X(\omega)$
2.2.1.11	Parseval's Relation	$x(t)$	$\int_{t=-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$.
	Other properties?	tx(t), even part, odd part	
		conjsym part, conjasym part	

Table 2.1: Properties of the Continuous Time Fourier Transform

This relation can be derived using the definition of the CTFS as follows,

$$\begin{aligned}
\int_{t=-\infty}^{\infty} |x(t)|^2 dt &= \int_{t=-\infty}^{\infty} x(t)x^*(t)dt \\
&= \int_{t=-\infty}^{\infty} x(t)\left(\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X^*(\omega)e^{-j\omega t}d\omega\right)dt \\
&= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X^*(\omega)\left(\int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt\right)d\omega \\
&= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X^*(\omega)\left(X(\omega)\right)d\omega \\
&= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X^*(\omega)X(\omega)d\omega \\
&= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} |X(\omega)|^2 d\omega.
\end{aligned}$$

Parseval's relation shows that the energy measured in the time-domain of a finite-energy signal is equal to the energy measured in the frequency domain through its CTFT representation.

2.2.2 CTFT Examples

Derivations of some of the signals in the Table 2.2.

2.3 Discrete-Fourier Series representation of DT periodic signals

In Section 2.1 we discussed the Fourier series representation as a means of building a large class of continuous time periodic signals from a set of simpler, harmonically related complex exponential signals. In this section, we consider the analogous notion of building a large class of periodic signals in discrete time from a set of simpler, harminically related complex exponential discrete time signals. An important difference between the continuous time Fourier series and what we will develop in this section as the discrete time Fourier series (DTFS), is that while the series used to construct periodic signals in continuous time is infinite, the series used to construct discrete time periodic signals is in fact a finite sum. This difference simplifies a number

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Continuous Time Signal	Continuous Time Fourier Transform
$x(t)$	$X(\omega) = \int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt$
$e^{-at}u(t),\ Real\{a\} > 0$	$\frac{1}{j\omega+a}$
$te^{-at}u(t),\ Real\{a\} > 0$	$\frac{1}{(j\omega+a)^2}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
1	$2\pi\delta(\omega)$
$\delta(t-T_0)$	$e^{-j\omega T_0}$
$\cos(\omega_0 t)$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin(\omega_0 t)$	$-j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$\frac{W}{\pi}\text{sinc}(\frac{Wt}{\pi}) = \begin{cases} \frac{sin(Wt)}{\pi t} & t \neq 0 \\ \frac{W}{\pi} & t = 0 \end{cases}$	$\begin{cases} 1, & \|\omega\| < W \\ 0, & \|\omega\| > W \end{cases}$
$\begin{cases} 1, & \|t\| < T \\ 0, & \|t\| > T \end{cases}$	$2\text{Tsinc}(\frac{\omega T}{\pi}) = \begin{cases} \frac{2sin(\omega T)}{\omega} & \omega \neq 0 \\ 2T & \omega = 0 \end{cases}$
more	more
more	more
more	more
more	more
more	more

Table 2.2: Continuous Time Fourier Transform Pairs

of issues that were delicate in the continuous case, such as notions of convergence, and existence of certain limits.

Mathematically, we represent a periodic discrete time signal, $x[n]$, as a signal whose value repeats at a fixed number of samples from the present. This interval, denoted N below, is called the "period" of the signal, and we express this relationship

$$x[n] = x[n+N], \text{ for all } n. \tag{2.20}$$

Equation (2.20) will, in general, be satisfied for a countably infinite number of possible values of N. The smallest, positive value of N for which Eq. (2.20) is satisfied, is called the "fundamental period" of the signal $x[n]$. Discrete time sinusoidal signals, such as

$$x[n] = \sin(\omega_0 n + \phi), \tag{2.21}$$

often enable us to relate the frequency of oscillation, ω_0 to a fundamental period, N. While analogous to their continuous time cousins, discrete time sinusoids need not always be periodic. While this may require a more careful notion of what is meant by discrete time "frequency," we will place this issue aside for the moment and consider how the period of a periodic sinusoid relates to the arguments of the sinusoidal function. This can again be computed by noting that sinusoidal functions are equal when their arguments are either equal or differ only through a multiple of 2π, i.e.

$$\begin{aligned} x[n] &= x[n+N] \\ \sin(\omega_0 n + \phi) &= \sin(\omega_0(n+N)+\phi) \\ \sin(\omega_0 n + \phi + 2k\pi) &= \sin(\omega_0(n+N)+\phi) \\ \sin(\omega_0(n+2k\pi/\omega_0)+\phi) &= \sin(\omega_0(n+N)+\phi) \end{aligned} \tag{2.22}$$

which yields the relationship

$$N = 2\pi k/\omega_0. \tag{2.23}$$

Depending on the value of ω_0, (2.23) may not provide an integer solution for N for any value of k. Note that only if ω_0/π is rational, will there be an integral solution to (2.23), for which the smallest integer value of N is the fundamental period associated with the discrete time frequency ω_0. In Figure (2.4), the two

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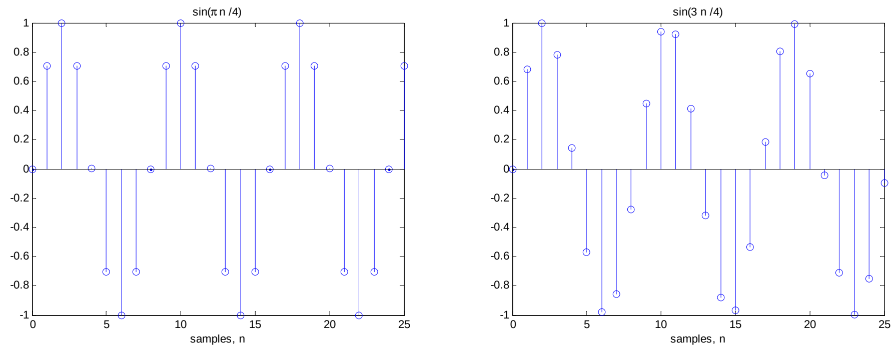

Figure 2.4: Examples of periodic and aperiodic sinusoidal signals $x[n] = \sin(\pi n/4)$ and $x[n] = \sin(3n/4)$.

sinusoidal signals $x[n] = \sin(\pi n/4)$ and $x[n] = \sin(3n/4)$ are shown. Note that the fundamental period of $N = 2\pi/(\pi/4) = 8$ can be readily seen from figure for the periodic signal $x[n] = \sin(\pi n/4)$. However, the aperiodic signal $x[n] = \sin(3n/4)$ does not exhibit periodicity for any value of n seen in the figure, and since the frequency argument of the sinusoid is not a rational multiple of π, we are guaranteed that no such integer period exists.

As in continuous time, any two periodic signals, $x[n]$ and $y[n]$ with the same period N can be added together to produce a new periodic signal of the same period, i.e.,

$$\begin{aligned} s[n] =& x[n] + y[n] \\ s[n+N] =& x[n+N] + y[n+N] = s[n+N]. \end{aligned}$$

We again consider how we might build-up a larger class of periodic signals from the basic building blocks of harmonically-related discrete time sinusoids. To extend our discussion to include complex-valued signals, we again employ Euler's relation to construct complex exponential signals of the form

$$\begin{aligned} x[n] =& e^{j(\omega_0 n + \phi)} \\ =& \cos(\omega_0 n + \phi) + j\sin(\omega_0 n + \phi) \end{aligned} \tag{2.24}$$

enabling us to write

$$x[n] = ce^{j\omega_0 n},$$

where, $c = e^{j\phi}$ is simply a complex constant whose effects on the sinusoidal nature of the signal have again been conveniently parked in front of the discussion. Complex-exponential signals of the form (2.24) may be periodic or aperiodic depending on whether or not ω_0/π is rational.

Analogous to the CTFS, we can explore the class of signals that can be constructed by such harmonically-related complex exponentials of the form

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{jk\omega_0 n}, \tag{2.25}$$

where $\omega_0 = 2\pi k/N$. Note that the summation in (2.25) only covers N terms, rather than the infinite sum in (2.6) for the CTFS. This is due to the fininte number of harmonically related complex exponentials that can be constructed with period N. Note that since the independent (time) variable in discrete time signals only takes on integer values, complex exponentials of frequency ω_0 are indistinguishable from those with frequency $\omega_0 + k2\pi$for any k, i.e.

$$e^{j\omega_0 n} = e^{j(\omega_0 + k2\pi)n}.$$

This result together with the fundamental period of N yields,

$$e^{j\frac{2\pi k}{N}n} = e^{j\frac{2\pi(k+N)}{N}n}.$$

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As a result, there are only N distinct complex exponential signals of period N. The resulting DTFS synthesis equation, written in terms of the fundamental period of the signal set becomes

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N}, \tag{2.26}$$

where $\omega_0 = 2\pi/N$ is the fundamental frequency of the periodic signal $x[n]$. The construction in (2.26) is referred to as the discrete-time Fourier series (DTFS) representation of $x[n]$ and (2.26) is often called the discrete-time Fourier series synthesis equation.

The Fourier series coefficients $X[k]$ can be obtained by multiplying (2.26) by $e^{-j2\pi kn/N}$ and summing over a period of duration N to obtain

$$\begin{aligned}
\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} &= \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{m=0}^{N-1} X[m]e^{j2\pi(m-k)n/N}\right) \\
\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} &= \frac{1}{N}\sum_{m=0}^{N-1} X[m]\left(\sum_{n=0}^{N-1} e^{j2\pi(m-k)n/N}\right).
\end{aligned}$$

To proceed, we need to evaluate the sum

$$\begin{aligned}
\sum_{n=0}^{N-1} e^{j2\pi(m-k)n/N} &= \begin{cases} \frac{1-e^{j2\pi(m-k)N/N}}{1-e^{j2\pi(m-k)/N}} & m \neq k \\ N & m = k \end{cases} \\
&= \begin{cases} \frac{1-e^{j2\pi(m-k)}}{1-e^{j2\pi(m-k)/N}} & m \neq k \\ N & m = k \end{cases} \\
&= \begin{cases} 0 & m \neq k \\ N & m = k, \end{cases}
\end{aligned}$$

which leads to the result

$$\begin{aligned}
\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} &= \frac{1}{N}\sum_{m=0}^{N-1} X[m]N\delta[m-k], \\
&= X[k],
\end{aligned}$$

by the sifting property of the Kronocker delta function. We can now return obtain the discretre-time Fourier series analysis equation,

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}. \tag{2.27}$$

Putting the synethesis and analysis equations together, we have the discrete-time Fourier series representation of a periodic signal $x[n]$ as

DT Fourier Series Representation of a Periodic Signal

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}, \text{ all } k \tag{2.28}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi kn}{N}}, \text{ all } n, \tag{2.29}$$

note that by convention, we define the signal $X[k]$ over all values of k, noting that due to the periodicity of the sequence $x[n]$ and of the signals $e^{j2\pi kn/N}$, the sequence $X[k]$will also be periodic with period N. In this

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derivation we used the following useful result for summation of a finite-length geometric series, which holds for any $r \neq 1$,

$$\sum_{k=m}^{n} r^k = \frac{r^m - r^{n+1}}{1-r}.$$

Example: DTFS of a Square Wave

Consider the periodic discrete time sequence of period $N = 8$ that satisfies

$$x[n] = \begin{cases} 1, & 0 \leq n < 4 \\ 0 & 4 \leq n < 8 \end{cases} \tag{2.30}$$

Using (2.27), we obtain,

$$\begin{aligned} X[k] &= \sum_{n=0}^{7} x[n] e^{-j2\pi kn/8} \\ &= \sum_{n=0}^{3} e^{-j2\pi kn/8} \\ &= \begin{cases} \frac{1-e^{-j2\pi k4/8}}{1-e^{-j2\pi k/8}}, & k \neq 0 \\ 4, & k = 0 \end{cases} \\ &= \begin{cases} \frac{1-e^{-j\pi k}}{1-e^{-j\pi k/4}}, & k \neq 0 \\ 4, & k = 0 \end{cases} \\ &= \begin{cases} \frac{e^{-j\pi k/2}(e^{j\pi k/2}-e^{-j\pi k/2})}{e^{-j\pi k/8}(e^{j\pi k/8}-e^{-j\pi k/8})}, & k \neq 0 \\ 4, & k = 0 \end{cases} \\ &= \begin{cases} e^{-j3\pi k/8} \frac{\sin(\pi k/2)}{\sin(\pi k/8)}, & k \neq 0 \\ 4, & k = 0. \end{cases} \end{aligned}$$

2.3.1 DT Fourier Series Properties

We have now been properly introduced to a method for building-up discrete-time periodic signals from a class of simple sinusoidal signals in (2.38) and a method for analysing the make-up of such periodic signals in terms of their constituent sinusoidal components in (2.37). Now that introductions are once again out of the way, we can explore some of the many useful properties of the DTFS representation.

2.3.1.1 Linearity

The DTFS can be viewed as a linear operation, in the following manner. When two signals $x[n]$ and $y[n]$ are each constructed from their constituent sinusoidal signals according to the DTFS synthesis equation (2.38), the linear combination of these signals, $z[n] = ax[n] + by[n]$, for a, b real-valued constants, can be readily obtained by combining the constituent sinusoidal signals through the same linear combination. More specifically, when $x[n]$ is a periodic signal with DTFS coefficients $X[k]$ and $y[n]$ is a periodic signal with CTFS coefficients $Y[k]$ then the signal $z[n] = ax[n] + by[n]$ has DTFS coefficients given by $Z[k] = aX[k] + bY[k]$. The linearity property of the DTFS can be compactly represented as follows

$$x[n] \overset{DTFS}{\longleftrightarrow} X[k], y[n] \overset{DTFS}{\longleftrightarrow} Y[k] \implies z[n] = ax[n] + by[n] \overset{DTFS}{\longleftrightarrow} aX[k] + bY[k].$$

This result can be readily shown by substituting $z[n] = ax[n] + by[n]$ into the summation in (2.37) and expanding the summation into the two separate terms, one for $X[k]$ and one for $Y[k]$.

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2.3.1.2 Time Shift

When a sinusoidal signal $x[n] = \sin[\omega_0 n]$ is shifted in time, the resulting signal $x[n-n_0]$ can be represented in terms of a simple phase shift of the origional sinusoidal signal, i.e. $x[n-n_0] = \sin(\omega_0(n-n_0)) = \sin(\omega_0 n + \phi)$,where $\phi = -\omega_0 n_0$. Periodic signals that can be represented using the DTFS contain many sinusoidal (or complex exponential) signals. When such periodic signals are delayed in time, each of the constituent sinusoidal components of the signal are delayed by the same amount, however this translates into a different phase shift for each component. This can be readily seen from the DTFS analysis equation 2.37, as follows. For the signal $y[n] = x[n-n_0]$, we have

$$
\begin{aligned}
Y[k] &= \sum_{n=0}^{N-1} x[n-n_0] e^{-j\frac{2\pi k}{N} n} \\
&= \sum_{m=N-n_0}^{N-1} x[m] e^{-j\frac{2\pi k}{N}(m+n_0)} + \sum_{m=0}^{N-1-n_0} x[m] e^{-j\frac{2\pi k}{N}(m+n_0)} \\
&= \sum_{m=0}^{N-1} x[m] e^{-j\frac{2\pi k}{N}(m+n_0)} \\
&= \sum_{m=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N} n_0} e^{-j\frac{2\pi k}{N} m} \\
&= X[k] e^{-j\frac{2\pi k}{N} n_0},
\end{aligned}
$$

where, the second line follows from the change of variable, $m = n - n_0$, and the third line follows from the periodicity of both the signal $x[n]$ and the signal $e^{-j2\pi kn/N}$with period N, and the last line follows from the definition of $X[k]$ after factoring the linear phase term $e^{-j2\pi kn_0/N}$ out of the sum. The time shift property of the DTFS can be compactly represented as follows

$$
x[n] \overset{DTFS}{\longleftrightarrow} X[k] \implies y[n] = x[n-n_0] \overset{DTFS}{\longleftrightarrow} X[k] e^{-j\frac{2\pi k}{N} n_0}.
$$

We see that a shift in time of a periodic signal corresponds to a modulation in frequency by a phase term that is linear with frequency with a slope that is proportional to the delay.

2.3.1.3 Frequency Shift

When a periodic signal $x[n]$ has a DTFS representation given by $X[k]$, a natural question that might arise is the what happens when the shifting that was discussed in section2.3.1.2 is applied to the DTFS representation, $X[k]$. Specifically, if a periodic signal $y[n]$ were known to have a DTFS representation given by $Y[k] = X[k-k_0]$, it is interesting to understand the relationship in the time-domain between $y[n]$ and $x[n]$. This can be readily seen through examination of the CTFS analysis equation,

$$
\begin{aligned}
Y[k] &= X[k-k_0] \\
&= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}(k-k_0)n} \\
&= \sum_{n=0}^{N-1} \left(x[n] e^{j\frac{2\pi}{N} k_0 n} \right) e^{-j\frac{2\pi}{N} kn}
\end{aligned}
$$

which leads to the relation

$$
x[n] \overset{DTFS}{\longleftrightarrow} X[k] \implies y[n] = x[n] e^{jk_0\omega_0 n} \overset{DTFS}{\longleftrightarrow} X[k-k_0],
$$

where $\omega_0 = \frac{2\pi}{N}$. We observe that a shift in the discrete time Fourier series coefficients by an integer amount k_0 corresponds to a modulation in the time domain signal $x[n]$ by a term whose frequency is proportional to the shift amount.

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2.3.1.4 Time Reversal

From the DTFS synthesis equation,

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi k}{N}n},$$

we see that by simply changing the sign of the time variable n, we obtain the relation

$$\begin{aligned} y[n] &= x[-n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi k}{N}n} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{-j\frac{2\pi(-k)}{N}n} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{-j\frac{2\pi(N-k)}{N}n} \\ &= \frac{1}{N}\sum_{m=0}^{N-1} X[N-m]e^{-j\frac{2\pi m}{N}n}, \end{aligned}$$

yielding the relation

$$x[n] \overset{DTFS}{\longleftrightarrow} X[k] \implies y[n] = x[-n] \overset{DTFS}{\longleftrightarrow} X[N-k],$$

i.e., changing the sign of the time axis corresponds to changing the sign of the DTFS frequency index, where, to keep the terms within the range from 0 to N, we add N to the index, which has no impact on their values, owing to the periodicity of the DTFS coefficients $X[k]$ with period N as a function of k.

2.3.1.5 Conjugate Symmetry

The effect of conjugating a complex-valued signal on its DTFS representation can be seen by simply conjugating the DTFS synthesis relation,

$$\begin{aligned} x[n] &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi k}{N}n} \\ x^*[n] &= \frac{1}{N}\left(\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi k}{N}n}\right)^* \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X^*[k]e^{-j\frac{2\pi k}{N}n} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X^*[k]e^{j\frac{2\pi(-k)}{N}n} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X^*[k]e^{j\frac{2\pi(N-k)}{N}n} \\ &= \frac{1}{N}\sum_{m=0}^{N-1} X^*[N-m]e^{j\frac{2\pi m}{N}n} \end{aligned}$$

yielding that

$$x[n] \overset{DTFS}{\longleftrightarrow} X[k] \implies x^*[n] \overset{DTFS}{\longleftrightarrow} X^*[N-k].$$

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When the periodic signal $x[n]$ is real valued, i.e. $x[n]$ only takes on values that are real numbers, then the DTFS exhibits a symmetry property. This arises directly from the definintion of the DTFS, and that real numbers equal their conjugates, i.e. $x[n] = x^*[n]$, such that

$$x[n] = x^*[n] \overset{DTFS}{\longleftrightarrow} X[k] \implies X[k] = X^*[N-k].$$

Note that when the signal is real-valued and is an even function of time, such that $x[n] = x[-n]$, then its DTFS is also real-valued and even, i.e. $X[k] = X^*[k] = X[-k] = X[N-k]$. It can be shown by similar reasoning that when the signal is periodic, real-valued, and an odd function of time, that the DTFS coefficients are purely imaginary and odd, i.e. $X[k] = -X^*[k] = -X[-k] = -X[N-k]$.

2.3.1.6 Products of Signals

When two periodic signals of the same period are multiplied in time, such that $z[n] = x[n]y[n]$, the resulting signal remains periodic with the same period, such that $z[n] = x[n]y[n] = x[n+N]y[n+N] = z[n+N]$. Hence, each of the three signals admit DTFS representations using the same set of harmonically related signals. We can observe the effect on the resulting DTFS representation through the analysis equation,

$$\begin{aligned} Z[k] &= \sum_{n=0}^{N-1} (x[n]y[n]) e^{-j\frac{2\pi k}{N}n} \\ &= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{m=0}^{N-1} X[m] e^{j\frac{2\pi m}{N}n} \right) y[n] e^{-j\frac{2\pi k}{N}n} \\ &= \frac{1}{N} \sum_{m=0}^{N-1} X[m] \left(\sum_{n=0}^{N-1} y[n] e^{-j\frac{2\pi(k-m)}{N}n} \right) \\ &= \frac{1}{N} \sum_{m=0}^{N-1} X[m] Y[k-m], \end{aligned}$$

where the periodicity of $Y[k]$ is used to determine values of $Y[k-m]$ for terms $k-m$ that fall outside the range of 0 to $N-1$. The relationship between the DTFS coefficients for $z[n]$ and those of $x[n]$ and $y[n]$ is seen to be a form of discrete convolution, called a periodic convolution, between the two sequences $X[k]$ and $Y[k]$,

$$x[n] \overset{DTFS}{\longleftrightarrow} X[k], y[n] \overset{DTFS}{\longleftrightarrow} Y[k] \implies z[n] = x[n]y[n] \overset{DTFS}{\longleftrightarrow} \frac{1}{N} \sum_{m=0}^{N-1} X[m]Y[k-m].$$

2.3.1.7 Convolution

A dual relationship to that of multiplication in time, is multiplication of DTFS coefficients. Specifically, when the two signals $x[n]$ and $y[n]$ are each periodic with period N, the periodic signal $z[n]$ of period N, whose DTFS representation is given by $Z[k] = X[k]Y[k]$ corresponds to a periodic convolution of the signals $x[n]$ and $y[n]$. This can be seen as follows,

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$$\begin{aligned}
z[n] &= \frac{1}{N}\sum_{k=0}^{N-1}\left(X[k]Y[k]\right)e^{j\frac{2\pi k}{N}n} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{N-1}x[m]e^{-j\frac{2\pi k}{N}m}\right)Y[k]e^{j\frac{2\pi k}{N}n} \\
&= \sum_{m=0}^{N-1}x[m]\left(\frac{1}{N}\sum_{k=0}^{N-1}Y[k]e^{j\frac{2\pi k}{N}(n-m)}\right) \\
&= \sum_{m=0}^{N-1}x[m]y[n-m]
\end{aligned}$$

where the summationin the last line is called periodic convolultion. This leads to the following property of the DTFS,

$$x[n]\overset{DTFS}{\longleftrightarrow}X[k], y[n]\overset{DTFS}{\longleftrightarrow}Y[k] \implies z[n]=\sum_{m=0}^{N-1}x[m]y[n-m]\overset{DTFS}{\longleftrightarrow}Z[k]=X[k]Y[k].$$

2.3.1.8 Parseval's relation

The energy containted within a period of a periodic signal can also be computed in terms of its CTFS representation using Parseval's relation,

$$x[n]\overset{DTFS}{\longleftrightarrow}X[k] \implies \sum_{n=0}^{N-1}|x[n]|^2=\frac{1}{N}\sum_{k=0}^{N-1}|X[k]|^2.$$

This relation can be derived using the definition of the DTFS as follows,

$$\begin{aligned}
\sum_{n=0}^{N-1}|x[n]|^2 &= \sum_{n=0}^{N-1}x[n]x^*[n] \\
&= \sum_{n=0}^{N-1}x[n]\left(\frac{1}{N}\sum_{k=0}^{N-1}X^*[N-k]e^{j\frac{2\pi k}{N}n}\right) \\
&= \frac{1}{N}\sum_{k=0}^{N-1}X^*[N-k]\left(\sum_{n=0}^{N-1}x[n]e^{j\frac{2\pi k}{N}n}\right) \\
&= \frac{1}{N}\sum_{k=0}^{N-1}X^*[N-k]\left(\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi(N-k)}{N}n}\right) \\
&= \frac{1}{N}\sum_{k=0}^{N-1}X^*[N-k]X[N-k] \\
&= \frac{1}{N}\sum_{m=0}^{N-1}|X[m]|^2.
\end{aligned}$$

Parseval's relation shows that the energy in a period of a periodic signal is equal to the sum of the energies contained within each of the harmonic components that make up the signal through the DTFS representation.

2.4 Discrete-time Fourier transform representation of DT signals

As with continuous-time signals, it is often convenient to represent discrete-time signals as a linear combination of simpler signals, or "basis signals". From continuous-time system theory, we know that complex

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exponential signals of the form e^{st} are a special class of signals called, "eigensignals" in that when placed as the input to a linear, time-invariant system, the output of the system will be of the form e^{st} scaled by a complex constant. As a result, such signals played an important role in the development of signal analysis and synthesis methods through the CT Fourier transform and Laplace transform. For discrete-time systems, we have that eigensignals of discrete-time linear-shift invariant systems include all signals that can be written in the form of a discrete-time complex exponential sequence, or z^n for all n and for any, possibly complex, z. By restricting the class of such signals to have unity magnitude, we arrive at the class of complex exponentials of the form $e^{j\omega n}$ for all n and for real-valued ω. These signals play a particularly important role in the analysis of discrete-time systems due to this eigenfunction property, which implies that the response of a linear shift-invariant system to a complex exponential input will be a complex exponential output of the same frequency with amplitude and phase determined by the system. For real-valued systems, i.e. systems with real-valued impulse responses, when the input is sinusoidal of a given frequency, the output will remain sinusoidal of the same frequency, again with amplitude and phase determined by the system. This important property of linear shift invariant systems makes the representation of signals in terms of complex exponentials extremely useful for studying linear system theory.

The discrete-time Fourier transform enables the construction of a wide class of signals from a superposition of complex exponentials. Through the eigenfunction property, the response of a linear shift invariant system to any signal in this class, i.e. any signal with a discrete-time Fourier transform, can be constructed by adding up the responses to each of the eigenfunctions that make up the original signal. By linearity of the system, the response of the system to a linear combination of complex exponentials will be given by the same linear combination of the responses to the complex exponentials. The eigenfunction property of LSI systems enables us to express very simply the response of the system to each of these complex exponentials.

The discrete-time Fourier transform, or DTFT, enables the representation of discrete-time sequences by a superposition of complex exponentials. Many sequences of interest can be represented by the following Fourier integral

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{j\omega n}d\omega, \tag{2.31}$$

where,

$$X_d(\omega) \quad = \quad \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{2.32}$$

is the discrete-time Fourier transform of the sequence $x[n]$. These two expressions comprise the Fourier representation of the sequence $x[n]$. Note that the DTFT, $X_d(\omega)$, is a complex-valued function of the real-valued variable ω, when the sum (2.32) exists. The integral corresponds to the inverse DTFT and represents the synthesis of the signal $x[n]$ from a superposition of signals of the form

$$\frac{1}{2\pi}X_d(\omega)e^{j\omega n}d\omega,$$

where we interpret the integral as the limit of a Riemann sum, i.e.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{j\omega n}d\omega = \lim_{\Delta\omega\to 0}\sum_{k=0}^{2\pi/\Delta\omega} X_d(-\pi + k\Delta\omega)e^{j\omega n}\Delta\omega.$$

The value of the DTFT, $X_d(\omega)$, determines the relative amount of each of the complex exponentials $e^{j\omega n}$ that is required to construct $x[n]$. The DTFT is referred to as Fourier analysis, as we analyze the composition of the signal in terms of the complex exponentials that make it up. The inverse DTFT is referred to as Fourier synthesis, as it can be viewed as synthesizing the signal from these basic components that make it up.

There is a strong similarity between the discrete-time Fourier transform and the z-transform for discrete-time signals that we will study in Chapter 5. This relationship is similar to that between the continuous-time Fourier transform and the Laplace transform for continuous-time signals. The more general Laplace transform of a continuous-time signal can be written

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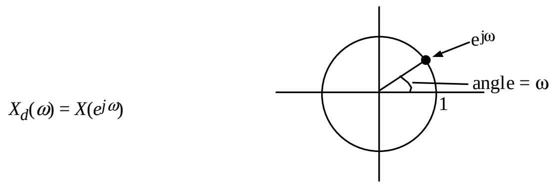

Figure 2.5: The DTFT viewed as evaluating the z-transform along the unit circle $z = e^{j\omega}$ in the z-plane.

$$X_L(s) = \int_{t=-\infty}^{\infty} x(t)e^{-st}dt, \tag{2.33}$$

when the integral exists. Substituting $s = j\omega$ into (2.33), yields the CTFT. So, for signals for which the CTFT exists, we can view the CTFT as a slice of the Laplace transform through the complex s-plane, along the imaginary axis. Just as the Fourier transform for continuous-time signals can be viewed as evaluating the Laplace transform along a specific curve, namely the imaginary axis in the s-plane, the DTFT can be viewed as evaluating the more general z-transform along a specific curve in the complex z-plane. The z-transform of a discrete-time sequence, given by,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \tag{2.34}$$

is the same as the DTFT for values of z evaluated for a particular slice of the complex z-plane. Specifically, the DTFT can be seen to be the same as the z-transform evaluated along a curve in the z-plane corresponding to the unit-circle, i.e.,

The DTFT exists as a regular function if and only if the region of convergence of the z-transform, the values of z for which the summation in (2.34) converges, includes the unit circle, i.e. $|z| = 1$. For the case of sinusoidal sequences, where $X(z)$ contains poles on the unit circle, $X_d(\omega)$ can be defined in terms of impulse distributions.

While for continuous-time signals, the notion of angular frequency is relatively well-defined, for discrete-time signals, we also refer to the variable ω in $X_d(\omega)$ as the digital frequency. Angular frequency in continuous-time is measured in Hz (cycles per second) or radians/sec. For discrete-time, angular frequency is measured in cycles per sample or radians per sample. In some textbooks the variable ω is used to represent analog frequency in the continuous-time Fourier transform. Here, we will use the variable ω to denote both continuous-time frequency and discrete-time frequency and the specific meaning will be clear by the context. For example, we will always refer to discrete-time Fourier transforms using the subscript "d" as in $X_d(\omega)$. Of course, we could use any variable for the DTFT and the continuous-time Fourier transform. When necessary, as in an expression relating a continuous-time frequency variable to an equivalent discrete-time frequency variable through sampling, we may use Ω to represent analog frequenecy and ω to represent digital frequency. This enables us to maintain clarity in our discussion and consistency with a number of other texts on the topic. It is important to recall that while continuous-time sinusoids have a fixed relationship between their frequency of oscillation and the period of the periodic time-domain waveform, discrete-time sinusoids may not be periodic at all. Recall that a signal of the form

$$x[n] = e^{j\omega n}$$

is only periodic if the following relation holds

$$x[n] = x[n + P].$$

Spefically, we must have that

$$e^{j(\omega_0 n + k2\pi)} = e^{j(\omega_0(n+P))}$$

which corresponds to requiring that

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$$\frac{2\pi}{\omega_0} = \frac{P}{k},$$

i.e., the digital frequency must be a rational multiple of π. This relationship will certainly only hold for a subset of all possible digital frequencies. Since the rational numbers are countable, and the real numbers are uncountable, this relationship does not hold almost everywhere in ω. That is, for practical purposes, almost any digital frequency you come up with, say by spinning a wheel and selecting the angle of the resulting position with respect to its starting position, will correspond to a complex exponential sequence that is not periodic. As a result, we rarely discuss the period of discrete-time complex exponentials and refer only to their digital frequency instead.

To demonstrate that the Fourier transform synthesis equation, or inverse DTFT, in fact inverts the DTFT, we can simply plug the definition of the DTFT into the synthesis equation as follows. From the DTFT synthesis equation, we have

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}\right)e^{j\omega n}d\omega \\ &= \sum_{m=-\infty}^{\infty} x[m]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-m)}d\omega\right) \\ &= \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \\ &= x[n], \end{aligned}$$

where we have used that

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-m)}d\omega &= \begin{cases} 1, & n = m \\ \frac{e^{j\pi(n-m)} - e^{-j\pi(n-m)}}{2\pi j(n-m)}, & n \neq m, \end{cases} \\ &= \begin{cases} 1, & n = m \\ \frac{(-1)-(-1)}{2\pi j(n-m)}, & n \neq m \end{cases} \\ &= \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases} \\ &= \delta[n-m]. \end{aligned}$$

2.4.1 Properties of the DTFT

A number of important properties of the DTFT can be derived in a manner similar to those for the DTFS. These are summarized at the end of this section in Table (2.5). Table (2.4) includes a number of DTFT pairs.

2.4.1.1 Linearity

The DTFT can be viewed as a linear operation, in the following manner. When two signals $x[n]$ and $y[n]$ satsify

$$x[n] \overset{DTFT}{\longleftrightarrow} X_d(\omega)$$

and

$$y[n] \stackrel{DTFT}{\longleftrightarrow} Y_d(\omega),$$

the linear combination of these signals, $z[n] = ax[n] + by[n]$, for a, b real-valued constants, can be readily obtained by combining the constituent complex exponential signals through the same linear combination. This is easily shown from the definition of the DTFT as follows

$$\begin{aligned} Z_d(\omega) &= \sum_{n=-\infty}^{\infty} (ax[n] + by[n])e^{-j\omega n} \\ &= a \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} + b \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} \\ &= aX_d(\omega) + bY_d(\omega). \end{aligned}$$

The linearity property of the DTFT can be compactly represented as follows

$$x[n] \stackrel{DTFT}{\longleftrightarrow} X_d(\omega), y[n] \stackrel{DTFT}{\longleftrightarrow} Y_d(\omega) \implies z[n] = ax[n] + by[n] \stackrel{DTFT}{\longleftrightarrow} aX_d(\omega) + bY_d(\omega).$$

2.4.1.2 Periodicity

The DTFT of every sequence is always periodic in that the following relation holds

$$X_d(\omega) = X_d(\omega + k2\pi),$$

for all integers k. The proof of this property lies in the periodicity of the complex exponential sequences $e^{j\omega n}$ that are used to construct each sequence with a DTFT as follows,

$$\begin{aligned} X_d(\omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+k2\pi)n} \\ &= X_d(\omega + k2\pi). \end{aligned}$$

This is different from the continuous-time Fourier transform, where we were interested in frequencies spanning an infinite range of real values. In contrast, in discrete-time, all digital frequencies can be captured in a single interval of length 2π. The reason for this periodicity stems directly from the observation that complex exponentials of the form $e^{j\omega n}$ are only unique over an interval of this range. That is the sequence $e^{j\omega n}$ is identical to the sequence $e^{j(\omega+2\pi)n}$. Since these two sequences have identical values, for all n, then the composition of $x[n]$ in terms of these sequences, i.e. the DTFT, must only require a single interval containing them. Since the DTFT is periodic with period 2π, the DTFT only needs to be specified over an interval of that length. It is often convenient to use the interval $-\pi \leq \omega \leq \pi$ so that the low frequencies are centered around $\omega = 0$. Note that since all frequencies that are multiples of 2π are indistinguishable, the low frequencies are also those centered around any multiple of 2π. Similarly, the highest digital frequency corresponds to $\omega = \pi$ as well as all odd multiples of π. We will return to this issue again when we discuss the discrete-time frequency response of linear shift-invariant systems.

2.4.1.3 Real and Imaginary Part Symmetries

For real-valued sequences $x[n]$, we have that the real-part of the DTFT is even, and the imaginary part is odd, i.e.,

The proof follows from trigonometric properties of the real and imaginary parts. Specifically, for real valued $x[n]$,we have

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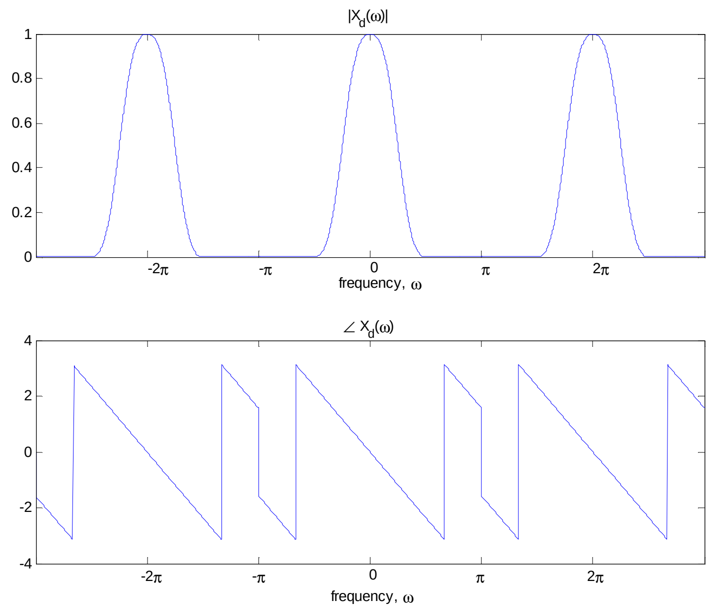

Figure 2.6: Magnitude and phase of an example DTFT.

$$\begin{aligned}
\Re\{X_d(\omega)\} &= \Re\{\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\} \\
&= \Re\{\sum_{n=-\infty}^{\infty} x[n](\cos(\omega n) + j\sin(\omega n))\} \\
&= \Re\{\sum_{n=-\infty}^{\infty} x[n](\cos(\omega n) + j\sin(\omega n))\} \\
&= \sum_{n=-\infty}^{\infty} x[n]\cos(\omega n) \\
&= \sum_{n=-\infty}^{\infty} x[n]\cos(-\omega n) \\
&= \Re\{X_d(-\omega)\}.
\end{aligned}$$

That the imaginary part of the DTFT is an odd function of ω, similarly follows from the antisymmetry of the sine function.

2.4.2 Magnitude and Phase Symmetries

For real-valued sequences $x[n]$, we have that the magnitude of the DTFT is an even function and the phase of the DTFT is an odd function, i.e.,

$$\begin{aligned}
|X(\omega)| &= |X_d(-\omega)| \\
\angle X(\omega) &= -\angle X_d(-\omega).
\end{aligned}$$

For example, $|X_d(\omega)|$ and $\angle X_d(\omega)$ might look as shown in Figure (2.6)

The proof of the magnitude and phase symmetries follows from the definition of the DTFT as below. From the definition of the DTFT we have

$$\begin{aligned}
|X_d(\omega)| &= \left[\left(\sum_{n=-\infty}^{\infty} x[n]\cos(\omega n)\right)^2 + \left(-\sum_{n=-\infty}^{\infty} x[n]\sin(\omega n)\right)^2\right]^{1/2} \\
&= \left[\left(\sum_{n=-\infty}^{\infty} x[n]\cos(\omega n)\right)^2 + \left(\sum_{n=-\infty}^{\infty} x[n]\sin(\omega n)\right)^2\right]^{1/2} \\
&= \left[\left(\sum_{n=-\infty}^{\infty} x[n]\cos(-\omega n)\right)^2 + \left(-\sum_{n=-\infty}^{\infty} x[n]\sin(-\omega n)\right)^2\right]^{1/2} \\
&= |X_d(-\omega)|,
\end{aligned}$$

and for the phase of the DTFT we have that

$$\begin{aligned}
\angle X_d(\omega) &= \arctan \frac{-\sum_{n=-\infty}^{\infty} x[n]\sin(\omega n)}{\sum_{n=-\infty}^{\infty} x[n]\cos(\omega n)} \\
&= \arctan \frac{\sum_{n=-\infty}^{\infty} x[n]\sin(-\omega n)}{\sum_{n=-\infty}^{\infty} x[n]\cos(-\omega n)} \\
&= -\arctan \frac{-\sum_{n=-\infty}^{\infty} x[n]\sin(-\omega n)}{\sum_{n=-\infty}^{\infty} x[n]\cos(-\omega n)} \\
&= -\angle X_d(\omega),
\end{aligned}$$

as desired. The last line above follows since arctan is an odd function of its argument.

2.4.2.1 Time Shift

As we have seen for both continuous-time and discrete-time periodic signals, when a sinusoidal signal is shifted in time, the resulting signal can be represented in terms of a simple phase shift of the origional sinusoidal signal. A discrete-time signal $x[n]$ that can be represented using the DTFT as a superposition of possibly infinitely many complex exponential signals of the form $e^{j\omega n}$,would necessarily have each of these constituent complex exponential signals delayed by the same fixed amount, which whould correspond to each of the complex exponential signals undergoing a different shift in the phase of their exponent. The resulting change in the DTFT of a discrete time signal $x[n]$ that is delayed by a fixed amount, i.e, $y[n] = x[n - n_0]$ can be derived as follows

$$\begin{aligned}
Y_d(\omega) &= \sum_{n=-\infty}^{\infty} x[n-n_0]e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} x[m]e^{-j\omega(m+n_0)} \\
&= \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega n_0}e^{-j\omega m} \\
&= X_d(\omega)e^{-j\omega n_0},
\end{aligned}$$

where, the second line follows from the change of variable, $m = n - n_0$. The time shift property of the DTFT can be compactly represented as follows

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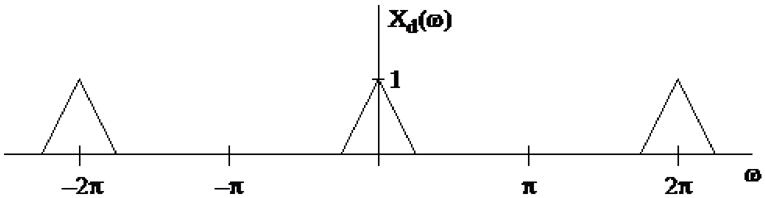

Figure 2.7: Sample DTFT $X_d(\omega)$ for the signal $x[n]$.

$$x[n] \overset{DTFT}{\longleftrightarrow} X_d(\omega) \implies y[n] = x[n-n_0] \overset{DTFT}{\longleftrightarrow} X_d(\omega)e^{-j\omega n_0}.$$

We see that a shift in time corresponds to a delay of each of the complex exponential components that make up the signal and that this delay, in turn corresponds to a shift in the phase of each of the frequency components by an amount that is linear with frequency with a slope that is proportional to the delay.

2.4.2.2 Modulation

When a signal $x[n]$ has a DTFT representation given by $X_d(\omega)$, we again are interested in how a shift in frequency would manifest itself in the time domain representation of the original signal. Specifically, if a signal $y[n]$ were known to have a DTFT representation given by $Y_d(\omega) = X_d(\omega - \omega_0)$, it is interesting to understand the relationship in the time-domain between $y[n]$ and $x[n]$. This can be readily seen through examination of the DTFT analysis equation,

$$\begin{aligned} Y_d(\omega) &= X_d(\omega-\omega_0) \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega-\omega_0)n} \\ &= \sum_{n=-\infty}^{\infty} \left(x[n]e^{j\omega_0 n}\right)e^{-j\omega n} \end{aligned}$$

which leads to the relation

$$x[n] \overset{DTFT}{\longleftrightarrow} X_d(\omega) \implies y[n] = x[n]e^{j\omega_0 n} \overset{DTFT}{\longleftrightarrow} X_d(\omega-\omega_0).$$

We observe that a shift in the discrete time Fourier transform by an amount ω_0 corresponds to a modulation in the time domain signal $x[n]$ by a term whose frequency is proportional to the shift amount. This property can be used together with linearity to determine the effect of modulation of a signal by a sinusoidal signal,

$$y[n] = \cos(\omega_0 n)x[n] = \frac{1}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right)x[n]$$

resulting in

$$x[n] \overset{DTFT}{\longleftrightarrow} X_d(\omega) \implies y[n] = x[n]\cos(\omega_0 n) \overset{DTFT}{\longleftrightarrow} \left[\frac{1}{2}X_d(\omega-\omega_0) + \frac{1}{2}X_d(\omega+\omega_0)\right].$$

Example:

If $x[n]$ has a DTFT as shown in Figure (2.7)

then $y[n] = e^{j\omega_0 n}x[n]$ would have the DTFT as shown in Figure (2.8).

Example

If $X_d(\omega)$ has the form shown in Figure (2.9),

then $Y_d(\omega)$, the DTFT of $y[n] = \cos(\omega_0 n)x[n]$ has the form shown in Figure (2.10)

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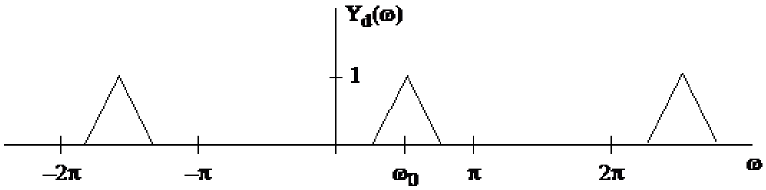

Figure 2.8: Resulting DTFT of $y[n] = e^{j\omega_0 n}x[n]$.

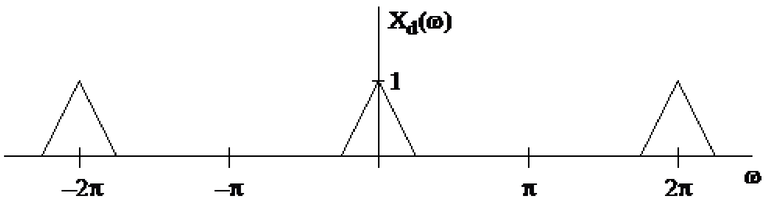

Figure 2.9: Example DTFT for $x[n]$.

2.4.2.3 Time Reversal

From the DTFT synthesis equation,

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{j\omega n}d\omega,$$

we see that by simply changing the sign of the time variable n, we obtain the relation

$$\begin{aligned} y[n] &= x[-n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{j\omega(-n)}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{j(-\omega)n}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(-\omega)e^{j\omega n}d\omega, \end{aligned}$$

yielding the relation

$$x[n] \overset{DTFT}{\longleftrightarrow} X_d(\omega) \implies y[n] = x[-n] \overset{DTFT}{\longleftrightarrow} X_d(-\omega),$$

i.e., changing the sign of the time axis corresponds to changing the sign of the DTFT frequency index, ω.

2.4.2.4 Conjugate Symmetry

The effect of conjugating a complex-valued signal on its DTFT representation can be seen by simply conjugating the DTFT synthesis relation,

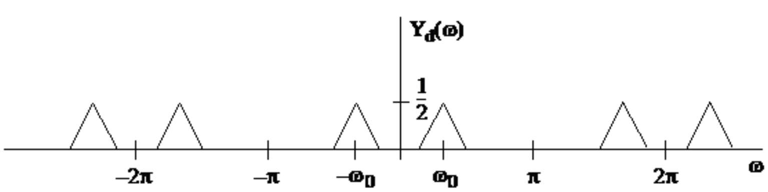

Figure 2.10: Resulting DTFT for $y[n]$.

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$$
\begin{aligned}
x[n] &= \int_{\omega=-\pi}^{\pi} X_d(\omega) e^{j\omega n} d\omega \\
x^*[n] &= \left(\int_{\omega=-\pi}^{\pi} X_d(\omega) e^{j\omega n} d\omega \right)^* \\
&= \int_{\omega=-\pi}^{\pi} X_d^*(\omega) e^{-j\omega n} d\omega \\
&= \int_{\omega=-\pi}^{\pi} X_d^*(\omega) e^{j(-\omega) n} d\omega \\
&= \int_{\omega=-\pi}^{\pi} X_d^*(-\omega) e^{j\omega n} d\omega
\end{aligned}
$$

yielding that

$$x[n] \overset{DTFT}{\longleftrightarrow} X_d(\omega) \implies x^*[n] \overset{DTFT}{\longleftrightarrow} X_d^*(-\omega).$$

When the periodic signal $x[n]$ is real valued, i.e. $x[n]$ only takes on values that are real numbers, then the DTFT exhibits additional symmetry. This arises directly from the definintion of the DTFT, and that real numbers equal their conjugates, i.e. $x[n] = x^*[n]$, such that

$$x[n] = x^*[n] \overset{DTFT}{\longleftrightarrow} X_d(\omega) \implies X_d(\omega) = X_d^*(-\omega).$$

Note that when the signal is real-valued and is an even function of time, such that $x[n] = x[-n]$, then its DTFT is also real-valued and even, i.e. $X_d(\omega) = X_d^*(-\omega) = X_d^*(\omega) = X_d(-\omega)$. It can be shown by similar reasoning that when the signal is real-valued and an odd function of time, that the DTFT is purely imaginary and odd, i.e. $X_d(\omega) = X_d^*(-\omega) = -X_d^*(\omega) = -X_d(-\omega)$.

2.4.2.5 Products of Signals

When two discrete-time signals that can each be represented by a DTFT are multiplied in time, such that $z[n] = x[n]y[n]$, the resulting signal also has a DTFT representation. We can observe the effect on the resulting DTFT representation through the analysis equation,

$$
\begin{aligned}
Z_d(\omega) &= \sum_{n=-\infty}^{\infty} (x[n]y[n]) e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{\nu=-\pi}^{\pi} X_d(\nu) e^{j\nu n} d\nu \right) y[n] e^{-j\omega n} \\
&= \frac{1}{2\pi} \int_{\nu=-\pi}^{\pi} X_d(\nu) \left(\sum_{n=-\infty}^{\infty} y[n] e^{-j(\omega-\nu)n} \right) d\nu \\
&= \frac{1}{2\pi} \int_{\nu=-\pi}^{\pi} X_d(\nu) Y_d(\omega-\nu) d\nu,
\end{aligned}
$$

where the periodicity of $Y_d(\omega)$ is used to determine values of $Y_d(\omega - \nu)$ for terms $\omega - \nu$ outside the range of $[-\pi, \pi]$. The relationship between the DTFTs of $z[n]$ and of $x[n]$ and $y[n]$ is seen to be a form of convolution, called a periodic convolution, between the two functions$X_d(\omega)$ and $Y_d(\omega)$,

$$x[n] \overset{DTFT}{\longleftrightarrow} X_d(\omega), y[n] \overset{DTFT}{\longleftrightarrow} Y(\omega) \implies z[n] = x[n]y[n] \overset{DTFT}{\longleftrightarrow} \frac{1}{2\pi} \int_{-\pi}^{\pi} X_d(\nu) Y_d(\omega-\nu) d\nu.$$

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2.4.2.6 Convolution

A dual relationship to that of multiplication of discrete-time signals in time, is the multiplication of their DTFT representations. Specifically, when two signals $x[n]$ and $y[n]$ have corresponding DTFT representations $X_d(\omega)$ and $Y_d(\omega)$, the signal that corresponds to the DTFT $Z_d(\omega) = X_d(\omega)Y_d(\omega)$ corresponds to a discrete-time convolution of the signals $x[n]$ and $y[n]$. This can be seen as follows,

$$\begin{aligned} z[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(X_d(\omega)Y_d(\omega)\right)e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}\right)Y_d(\omega)e^{j\omega n}d\omega \\ &= \sum_{m=-\infty}^{\infty} x[m]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} Y_d(\omega)e^{j\omega(n-m)}d\omega\right) \\ &= \sum_{m=-\infty}^{\infty} x[m]y[n-m]. \end{aligned}$$

This leads to the following property of the DTFT,

$$x[n] \overset{DTFT}{\longleftrightarrow} X_d(\omega), y[n] \overset{DTFT}{\longleftrightarrow} Y_d(\omega) \implies z[n] = \sum_{m=-\infty}^{\infty} x[m]y[n-m] \overset{DTFT}{\longleftrightarrow} Z_d(\omega) = X_d(\omega)Y_d(\omega).$$

2.4.2.7 Parseval's relation

A particularly useful relationship between the energy of a sequence in the time-domain and the energy contained in its Fourier transform is captured by Parseval's relation. Since sequences with convergent DTFTs are square summable, they have finite energy and we have that

$$x[n] \overset{DTFT}{\longleftrightarrow} X_d(\omega) \implies \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X_d(\omega)|^2 d\omega.$$

This relation can be derived using the definition of the DTFT as follows,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |x[n]|^2 &= \sum_{n=-\infty}^{\infty} x[n]x^*[n] \\ &= \sum_{n=-\infty}^{\infty} x[n]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{j\omega n}d\omega\right)^* \\ &= \sum_{n=-\infty}^{\infty} x[n]\frac{1}{2\pi}\int_{-\pi}^{\pi} X_d^*(\omega)e^{-j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d^*(\omega)\left(\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right)d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d^*(\omega)X_d(\omega)d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X_d(\omega)|^2 d\omega. \end{aligned}$$

Example:

Parseval's relation can be used to compute the energy of a signal in the time domain or the frequency domain. As a result, one of these is often simpler than the other. For example,

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consider the sequence $x[n] = u[n] - u[n-10]$. A rather complicated integral can be reduced to a simple sum by noting the transform pair

$$u[n] - u[n-10] \stackrel{DTFT}{\longleftrightarrow} \frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

and using Parseval's relation as follows:

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\sin(5\omega/2)}{\sin(\omega/2)}\right|^2 d\omega &= \sum_{n=0}^{9}(1)^2 \\ &= 10 \end{aligned}$$

Examples of DTFT

We continue our discussion of the discrete-time Fourier transform by considering a few simple examples.

Example

Consider the following sequence containing two non-zero samples,

$$x[n] = \delta[n] + \delta[n-1].$$

The discrete-time Fourier transform of this sequence can be computed directly from the definition of the DTFT as

$$X_d(\omega) = 1 + e^{-j\omega}.$$

The magnitude of the discrete-time Fourier transform can be easily computed as

$$\begin{aligned} |X_d(\omega)|^2 &= |1 + e^{-j\omega}|^2 \\ &= |1 + \cos(\omega) - j\sin(\omega)|^2 \\ &= |1 + \cos(\omega)|^2 + |\sin(\omega)|^2 \\ &= 1 + 2\cos(\omega) + \cos^2(\omega) + \sin^2(\omega) \\ &= 2 + 2\cos(\omega). \end{aligned}$$

This result could also have been obtained by noting that when two exponential terms (or one exponential term and one constant term) have the same magnitude, by factoring out a common phase factor, a sinusoid can be constructed as follows,

$$\begin{aligned} |X_d(\omega)|^2 &= |1 + e^{-j\omega}|^2 \\ &= |e^{-j\omega/2}(e^{j\omega/2} + e^{-j\omega/2})|^2 \\ &= \left|e^{-j\omega/2}\right|^2 \left|e^{j\omega/2} + e^{-j\omega/2}\right|^2 \\ &= \left|e^{j\omega/2} + e^{-j\omega/2}\right|^2 \\ &= |2\cos(\omega/2)|^2 \\ &= 4\cos^2(\omega/2) \\ &= 2 + 2\cos(\omega), \end{aligned}$$

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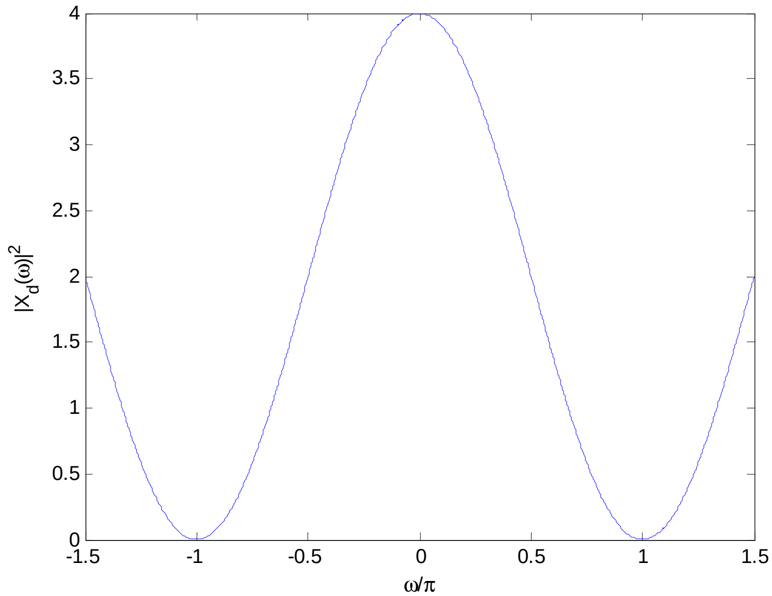

Figure 2.11: Discrete-time Fourier transform magnitude squared, $|X_d(\omega)|^2 = 2 + 2\cos(\omega)$.

where, in the last line, the trigonometric identity that $\cos^2(x) = (1+\cos(2x))/2$. The magnitude squared of the discrete-time Fourier transform is shown in Figure (2.11).

Since $|X_d(\omega)|^2$ is both periodic with period 2π and symmetric around the origin, $|X_d(\omega)|^2$ is completely determined by its values on the interval $0 \le \omega \le \pi$. This is similarly true for $\angle X_d(\omega)$. Because of this, when $x[n]$ is real (so that $|X_d(\omega)|$ and $\angle X_d(\omega)$ have even and odd symmetry, respectively) we will often plot them on just the interval $0 \le \omega \le \pi$.

To find $\angle X_d(\omega)$ in this example, we write

$$\begin{aligned} X_d(\omega) &= 1 + e^{-j\omega} \\ &= e^{-j\omega/2}(e^{j\omega/2} + e^{-j\omega/2}) \\ &= e^{-j\omega/2} 2\cos(\omega/2), \end{aligned} \tag{2.35}$$

Now, since $\cos(\omega/2) \ge 0$ for $-\pi < \omega < \pi$, (2.35) expresses $X_d(\omega)$ in polar form, so that $\angle X_d(\omega) = -\omega/2$, $-\pi < \omega < \pi$. The phase is plotted in Figure (2.12).

Notice that the phase of $X_d(\omega)$ is an odd function. Also note that for $\omega > \pi$, the expression $\angle X_d(\omega) = -\omega/2$ is not valid, however we simply use our knowledge that the DTFT is periodic with period 2π. While it is clear from this example that the discrete-time Fourier transform is periodic in the variable ω with period 2π, the DTFT is, in fact, always periodic with period 2π, as shown in Section 2.4.1.2.

Example

Consider the sequence $x[n] = \delta[n-1] - \delta[n+1]$. For this sequence, we will plot the magnitude $|X_d(\omega)|$ and phase $\angle X_d(\omega)$. For the magnitude, we have

$$\begin{aligned} |X_d(\omega)| &= \left|e^{-j\omega} - e^{j\omega}\right| \\ &= |-2j\sin(\omega)| \\ &= |2\sin(\omega)|, \end{aligned}$$

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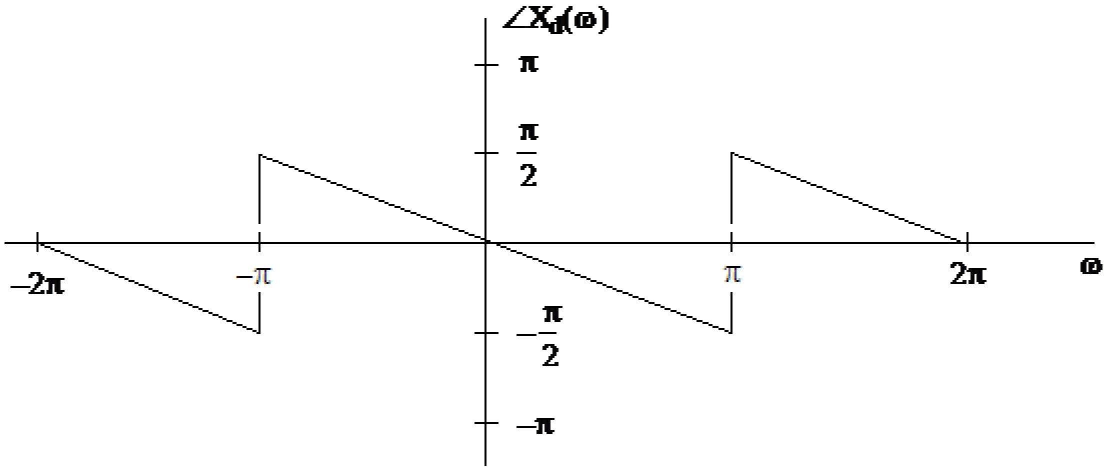

Figure 2.12: Example DTFT phase for $x[n] = \delta[n] + \delta[n-1]$.

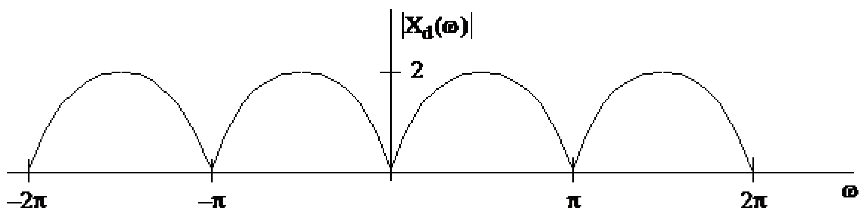

Figure 2.13: Magnitude of the DTFT for $x[n] = \delta[n-1] - \delta[n+1]$.

which can be plotted as shown in Figure (2.13)

Note that $|X_d(\omega)|$ is again an even function of ω, and its appearance is completely specified by on just the interval $0 \leq \omega \leq \pi$. The phase of $X_d(\omega)$ is found by noting

$$\begin{aligned} X_d(\omega) &= \begin{cases} -2j\sin(\omega), & \{\omega : \sin(\omega) > 0\} \\ 2j\sin(\omega), & \{\omega : \sin(\omega) < 0\} \end{cases} \\ &= \begin{cases} e^{-j\pi/2} 2\sin(\omega), & 0 < \omega < \pi \\ e^{j\pi/2}|2\sin(\omega)|, & -\pi < \omega < 0. \end{cases} \end{aligned}$$

Both the top and bottom lines within the bracket are written in polar form, since $\sin(\omega) > 0$ for $0 < \omega < \pi$, and $|\sin(\omega)| > 0$. Thus,

$$\angle X_d(\omega) = \begin{cases} -\frac{\pi}{2}, & 0 < \omega < \pi \\ \frac{\pi}{2}, & -\pi < \omega < 0 \end{cases}$$

which can be plotted as in Figure (2.14).

Notice that $\angle X_d(\omega)$ is an odd function of the variable ω. This is again due to the symmetry properties of the DTFT for real-valued $x[n]$. It is important to recall that these symmetry properties of and $\angle X_d(\omega)$ hold only for real-valued sequences $x[n]$. If $x[n]$ were not real-valued, then this symmetry will not be present, as shown in the next example.

Example

Consider the sequence $x[n] = \delta[n] + j\delta[n-1]$. For this sequence, we have the following discrete-time Fourier transform,

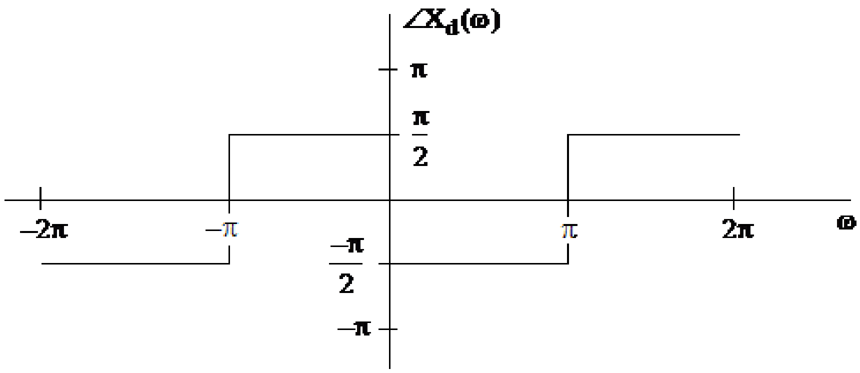

Figure 2.14: DTFT phase example.

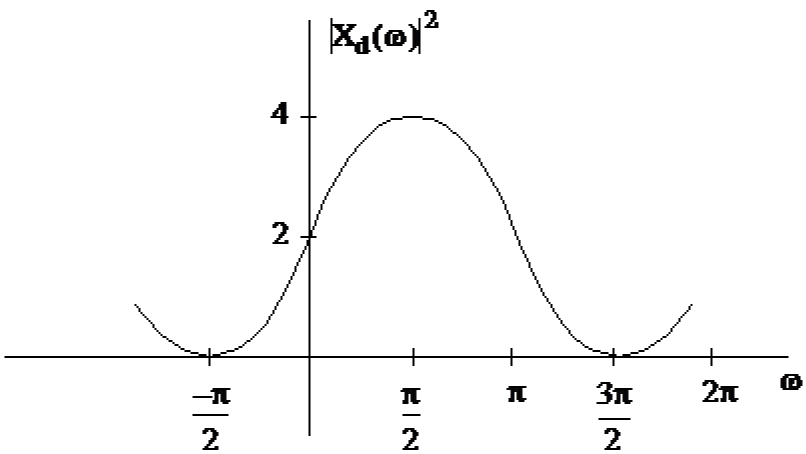

Figure 2.15: DTFT magnitude squared for the sequence $x[n] = \delta[n] + j\delta[n-1]$.

$$X_d(\omega) = 1 + je^{-j\omega},$$

which has a corresponding DTFT magnitude squared

$$\begin{aligned} |X_d(\omega)|^2 &= (1 + je^{-j\omega})(1 - je^{j\omega}) \\ &= 1 - je^{j\omega} + je^{-j\omega} + 1 \\ &= 2 + 2\sin(\omega). \end{aligned}$$

The DTFT magnitude squared is shown in Figure (2.15).

Here, we see that $|X_d(\omega)| = |X_d(-\omega)|$ does not hold.

Example

Consider the following sequence $x[n] = a^n u[n]$, with a real-valued and $|a| < 1$. For this signal, we have

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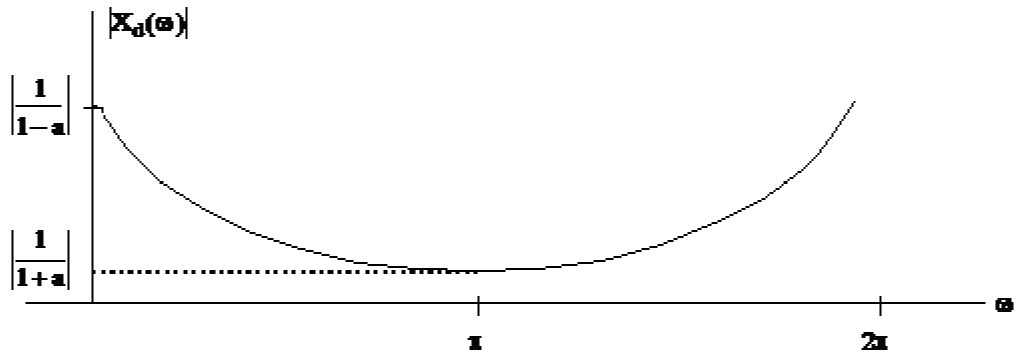

Figure 2.16: DTFT Magnitude Squared for $x[n] = a^n u[n]$.

$$\begin{aligned} X_d(\omega) &= \sum_{n=-\infty}^{\infty} a^n e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} (ae^{-j\omega})^n \\ &= \frac{1}{1 - ae^{-j\omega}} \end{aligned}$$

The DTFT magnitude squared is given by

$$\begin{aligned} |X_d(\omega)| &= \frac{1}{(1 - ae^{-j\omega})(1 - ae^{j\omega})} \\ &= \frac{1}{1 + a^2 - 2a\cos(\omega)} \end{aligned}$$

which for $0 < a < 1$, the magnitude squared would appear as depicted in Figure (2.16).

Example

We next consider a sinusoidal input, $x[n] = \cos(\omega_0 n)$ for all n. As we will see in Chapter 5, the z-transform for this sequence is undefined for all z, including z on the unit circle, where it coincides with the discrete-time Fourier transform. However, we are willing to extend the notion of existence of the discrete-time Fourier transform to include such signals with the aid of impulse distributions. Similar to their continuous-time counterparts, we can define the DTFT of discrete-time sinusoidal signals in terms of impulses. As such, we can define the DTFT of such a signal to satisfy

$$\begin{aligned} X_d(\omega) &= \sum_{n=-\infty}^{\infty} \cos(\omega_0 n) e^{-j\omega n} \\ &= \pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)), \quad |\omega| < \pi, \end{aligned}$$

in the sense that the inverse discrete-time Fourier transform of this distribution would yield the original signal $x[n]$. Here, $X_d(\omega)$ is a distribution, not a function, so the DTFT does not really exists in the normal sense, and the summation defining the DTFT does not converge in any meaningful sense to any function. However, if we use this distribution as the operational DTFT of the sequence, then taking its inverse DTFT would yield

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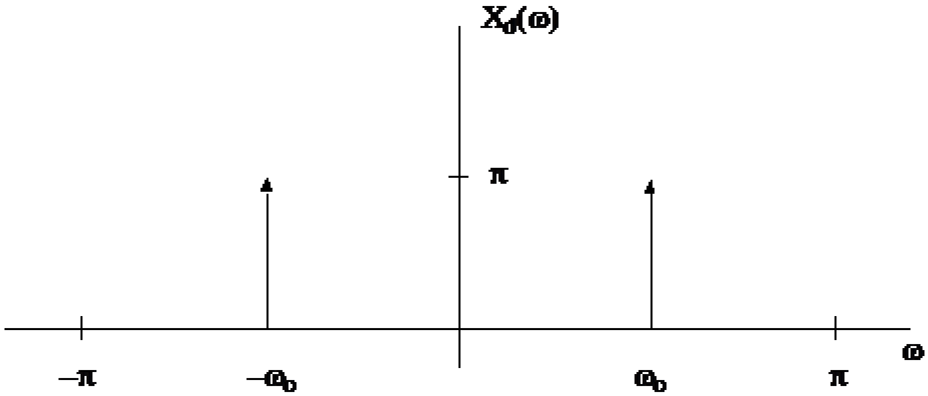

Figure 2.17: Depiction of the impulse distribution for the DTFT of $x[n] = \cos(\omega_0 n)$.

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{j\omega n}d\omega &= \frac{\pi}{2\pi}\int_{-\pi}^{\pi}(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))e^{j\omega n}d\omega \\ &= \frac{1}{2}(e^{j\omega_0 n}+e^{-j\omega_0 n}) \\ &= \cos(\omega_0 n).\end{aligned}$$

If we graphically depict the distribution $X_d(\omega)$, using our notation for representing impulse distributions, we would obtain the sketch in Figure (2.17).

While an impulse distribution cannot be considered a function, nor can it be considered a proper limit of a sequence of functions, we could approximate the above figure using tall, narrow rectangles around the frequencies of interest, i.e. in place of $\delta(\omega-\omega_0)$ and $\delta(\omega+\omega_0)$. This would correspond to the discrete-time Fourier transform of an approximation to $\cos(\omega_0 n)$. Additionally, the quality of the approximation improves as the rectangles get narrower and taller. This is explored further in the next Example.

Example

Consider the finite-length sequence $x[n]$ described below,

$$x[n] = \begin{cases}\cos(\omega_0 n), & 0 \leq n \leq N-1 \\ 0 & \text{otherwise.}\end{cases}$$

The sequence is finite-length, in that it has a finite number, N, of non-zero samples. This sequence can be constructed as a windowed version of the original infinite-length sequence $\cos(\omega_0 n)$. The DTFT of the sequence can be written,

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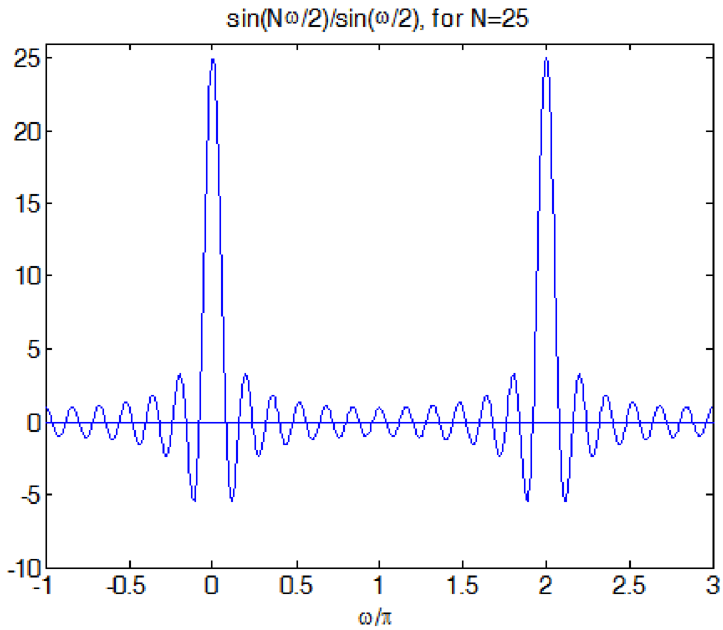

Figure 2.18: Periodic sinc function, $\frac{\sin(N\omega/2)}{\sin(\omega/2)}$ for $N = 25$. Note that the first zero-crossing occurs at $\omega = 2\pi/N$.

$$
\begin{aligned}
X_d(\omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\
&= \sum_{n=0}^{N-1} \cos(\omega_0 n)e^{-j\omega n} \\
&= \sum_{n=0}^{N-1} \frac{1}{2}[e^{j\omega_0 n} + e^{-j\omega_0 n}]e^{-j\omega n} \\
&= \sum_{n=0}^{N-1} \frac{1}{2}e^{j\omega_0 n}e^{-j\omega n} + \sum_{n=0}^{N-1} \frac{1}{2}e^{-j\omega_0 n}e^{-j\omega n} \\
&= \frac{1}{2}\frac{1-e^{-j(\omega-\omega_0)N}}{1-e^{-j(\omega-\omega_0)}} + \frac{1}{2}\frac{1-e^{-j(\omega+\omega_0)N}}{1-e^{-j(\omega+\omega_0)}} \\
&= \frac{1}{2}\frac{e^{-j(\omega-\omega_0)N/2}\left(e^{j(\omega-\omega_0)N/2}-e^{-j(\omega-\omega_0)N/2}\right)}{e^{-j(\omega-\omega_0)/2}\left(e^{j(\omega-\omega_0)/2}-e^{-j(\omega-\omega_0)/2}\right)} + \frac{1}{2}\frac{e^{-j(\omega+\omega_0)N/2}\left(e^{j(\omega+\omega_0)N/2}-e^{-j(\omega+\omega_0)N/2}\right)}{e^{-j(\omega+\omega_0)/2}\left(e^{j(\omega+\omega_0)/2}-e^{-j(\omega+\omega_0)/2}\right)} \\
&= \frac{1}{2}e^{-j(\omega-\omega_0)(N-1)/2} \underbrace{\frac{\sin(N(\omega-\omega_0)/2)}{\sin((\omega-\omega_0)/2)}}_{\text{periodic sinc centered at} \omega=\omega_0} + \frac{1}{2}e^{-j(\omega+\omega_0)(N-1)/2} \underbrace{\frac{\sin(N(\omega+\omega_0)/2)}{\sin((\omega+\omega_0)/2)}}_{\text{periodic sinc centered at} \omega=-\omega_0}
\end{aligned}
$$

Note that this expression contains two terms; one term corresponding to a ratio of sin expressions, multiplied by a linear phase term, and another corresponding to a similar ratio of sin expressions and a similar linear phase term. Each of these terms, corresponds to a periodic sinc function, centered at the corresponding positive and negative frequencies of the original cosine expression. The periodic sinc function is simply the DTFT of a length-N sequence of one's, and is depicted in Figure (2.18).

For large N, the main lobe in the periodic sinc becomes narrow and large in amplitude so that

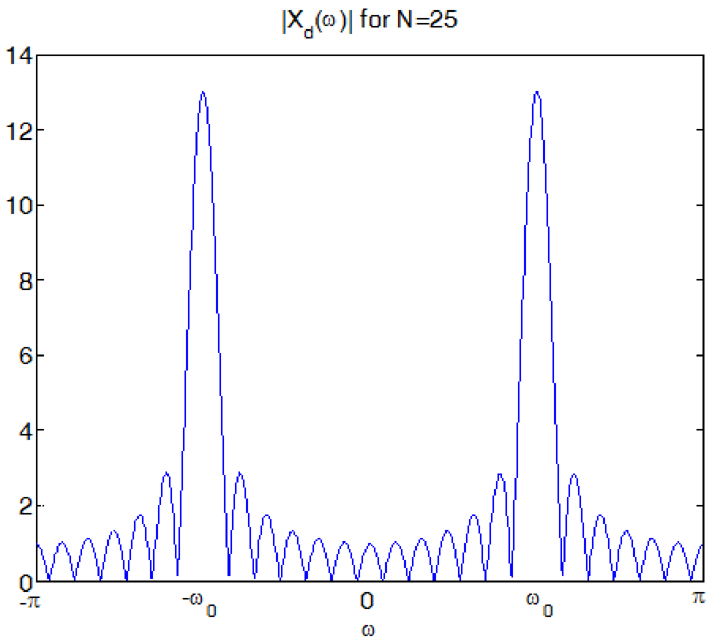

Figure 2.19: The magnitude of the DTFT of the sequence $x[n] = \cos(\omega_0 n)$, for $0 \leq n \leq 24$ and $x[n] =$ 0,elsewhere.

the two terms in the DTFT of the windowed cosine sequence do not overlap much and we have that

$$|X_d(\omega)| \approx \frac{1}{2}\left|\frac{\sin(N(\omega-\omega_0)/2)}{\sin((\omega-\omega_0)/2)}\right| + \frac{1}{2}\left|\frac{\sin(N(\omega+\omega_0)/2)}{\sin((\omega+\omega_0)/2)}\right|$$

This relation is expressed in Figure (2.19), again for the case of $N = 25$.

Now, as N becomes large, this figure begins to resemble, in some sense, the figure containing two impulses. Similarly, as N becomes large, the windowed (truncated) cosine sequence becomes a better approximation of the infinite-length cosine sequence.

2.5 Discrete Fourier Transform representation of finite-length DT signals

In Section 2.3 we discussed the discrete Fourier series representation as a means of building a large class of discrete-time periodic signals from a set of simpler, harminically related discrete-time complex exponential signals. In this section, we introduce the discrete Fourier transform (DFT) as an analogous notion of building a large class of finite-length signals from a set of simpler, harminically related finite-length complex exponential discrete time signals. An important difference between the discrete-time Fourier series and what we will develop in this section as the discrete Fourier transform, is that while the periodic signals and the complex exponential signals used to construct them in the case of the DTFS were defined for all n, the signals for which we consider a DFT representation are finite in length and are therefore only defined for a finite interval of the time axis, n. Since this is a subtle difference, we are able to capitalize on all of the development of the DTFS. By considering a finite-length signal defined only on the interval $0 \leq n \leq N-1$, as one period of an infinite-length periodic signal defined for all n, we can directly map the DTFS into the DFT for our purposes. Mathematically, if a signal $x[n]$ is defined only on the interval $0 \leq n \leq N-1$, then by considering the periodic signal $\tilde{x}[n]$, defined as follows

Section	DTFT Property	Discrete Time Signal	Discrete Time Fourier Transform
	Definition	$x[n]$	$X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
2.2.1.1	Linearity	$z[n] = ax[n] + by[n]$	$Z_d(\omega) = aX_d(\omega) + bY_d(\omega)$
2.4.1.2	Periodicity	$x[n]$	$X_d(\omega) = X_d(\omega + k2\pi)$
2.4.1.3,2.4.2	Real Part Symmetry	$x[n]$ real valued	$\Re\{X_d(\omega)\} = \Re\{X_d(-\omega)\}$
2.4.1.3,2.4.2	Imaginary Part Symmetry	$x[n]$ real valued	$\Im\{X_d(\omega)\} = -\Im X_d(-\omega)$
2.4.1.3,2.4.2	Magnitude Symmetry	$x[n]$ real valued	$\|X_d(\omega)\| = \|X_d(-\omega)\|$
2.4.1.3,2.4.2	Phase Symmetry	$x[n]$ real valued	$\angle X_d(\omega) = -\angle X_d(-\omega)$
2.4.1.3,2.4.2	Conjugate Symmetry	$x[n]$ real valued	$X_d(\omega) = X_d^*(-\omega)$
2.4.2.1	Time Shift	$y[n] = x[n-d]$	$Y_d(\omega) = X_d(\omega)e^{-j\omega d}$
2.4.2.2	Modulation	$y[n] = x[n]e^{j\omega_0 n}$	$Y_d(\omega) = X_d(\omega - \omega_0)$
2.4.2.3	Time-Reversal	$y[n] = x[-n]$	$Y_d(\omega) = X_d(-\omega)$
2.4.2.4	Conjugation	$y[n] = x^*[n]$	$Y_d(\omega) = X^*(-\omega)$
2.4.2.5	Product of Signals	$z[n] = x[n]y[n]$	$Z_d(\omega) = \frac{1}{2\pi}\int_{\nu=-\pi}^{\pi} X_d(\nu)Y_d(\omega-\nu)d\nu$
2.4.2.6	Convolution	$z[n] = \sum_{m=-\infty}^{\infty} x[m]y[n-m]$	$Z_d(\omega) = X_d(\omega)Y_d(\omega)$
2.4.2.7	Parseval's Relation	$x[n]$	$\sum_{n=-\infty}^{\infty} \|x[n]\|^2 = \frac{1}{2\pi}\int_{\omega=-\pi}^{\pi} \|X_d(\omega)\|^2.$

Table 2.3: Properties of the Discrete Time Fourier Transform

Discrete Time Signal	Discrete Time Fourier Transform
$x[n]$	$X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
$a^n u[n],\ \|a\| < 1$	$\frac{1}{1-ae^{-j\omega}}$
$na^n u[n],\ \|a\| < 1$	$\frac{ae^{-j\omega}}{(1-ae^{-j\omega})^2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2k\pi)$
1	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2k\pi)$
$\delta[n]$	1
$\cos(\omega_0 n)$	$\sum_{k=-\infty}^{\infty} \pi[\delta(\omega - \omega_0 + 2k\pi) + \delta(\omega + \omega_0 + 2k\pi)]$
$\sin(\omega_0 n)$	$-j\pi\sum_{k=-\infty}^{\infty} [\delta(\omega - \omega_0 + 2k\pi) - \delta(\omega + \omega_0 + 2k\pi)]$
$\frac{\omega_0}{\pi}\text{sinc}\left(\frac{\omega_0 n}{\pi}\right) = \begin{cases} \frac{sin(\omega_0 n)}{\pi n} & n \neq 0 \\ \frac{\omega_0}{\pi} & n = 0 \end{cases}$	$\begin{cases} 1, & \|\omega\| \leq \omega_0 \\ 0, & \omega_0 < \|\omega\| \leq \pi \end{cases}$
$\begin{cases} 1, & 0 \leq n < N \\ 0, & \text{otherwise} \end{cases}$	$\begin{cases} \frac{\sin(N\omega/2)}{\sin(\omega/2)} & \omega \neq 0 \\ N & \omega = 0 \end{cases}$

Table 2.4: Discrete Time Fourier Transform Pairs

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$$\tilde{x}[n] = \sum_{k=-\infty}^{\infty} x[n + rN], \tag{2.36}$$

then over the interval from $0 \leq n \leq N-1$, we have that $x[n] = \tilde{x}[n]$.

The Fourier series coefficients $\tilde{X}[k]$ for this periodic signal can be obtained by

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi kn/N}.$$

While this expression is valid for all k, we only require one period of $\tilde{X}[k]$, i.e. $0 \leq k \leq N-1$, for the inverse DTFS relation to reconstruct $\tilde{x}[n]$. Putting the synethesis and analysis equations together, we have the discrete-time Fourier series representation of a periodic signal $x[n]$ as

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi kn}{N}},$$

which is again valid for all n, since the signal $\tilde{x}[n]$ is periodic and defined for all n. However, if we are only interested in finite-length signals, then viewing them as a single period of an infinite-length periodic signal as in (2.36), we can use the DTFS to both analyze and reconstruct finite-length signals from the first period of the underlying infinite-length periodic signals $\tilde{x}[n]$ and $\tilde{X}[k]$. Specifically, we can define the discrete Fourier transform of a finite-length signal, defined only over an interval of length N samples as

Discrete Fourier Transform Representation of a Finite-Length Signal

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}, 0 \leq k \leq N-1, \tag{2.37}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi kn}{N}}, 0 \leq n \leq N-1. \tag{2.38}$$

2.5.1 Discrete Fourier Transform Properties

We can now explore some of the many useful properties of the DFT representation, noting that these follow directly from the properties of the DTFS.

2.5.1.1 Sampling Property

While DFT is related to the DTFS of the periodic signal $\tilde{x}[n] = \sum_{k=-\infty}^{\infty} x[n + kN]$, it can also be shown to be related to the DTFT of the infinite length signal $x_{zp}[n] = x[n]$, $0 \leq n \leq N-1$,and $x_{zp}[n] = 0$, outside this region. The subscript "zp" stands for "zero-padding", where the infinite length signal $x_{zp}[n]$, can be viewed as padding the finite length signal $x[n]$ with zeros outside the interval $0 \leq n \leq N-1$, over which it is defined. By observing the similarity between the DTFT and DFT representations,

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n} = \sum_{n=-\infty}^{\infty} x_{zp}[n] e^{-j\omega n}\Big|_{\omega=\frac{2\pi k}{N}} = X_d(\omega)\Big|_{\omega=\frac{2\pi k}{N}},$$

where $X_d(\omega)$ here refers to the DTFT of the infinite-length sequence $x_{zp}[n]$. We see that the DFT of the finite-length signal $x[n]$ can be viewed as a set of N evenly-spaced samples of the DTFT of the zero-padded infinite-length signal $x_{zp}[n]$ taken at samples $\omega_k = 2\pi k/N$, for $0 \leq k \leq N-1$. This can be seen pictorially in Figure (2.20).

Note that the last DFT sample, $X[N-1]$ does not correspond to a sample taken at $\omega = 2\pi$, but rather to the left of 2π, at $\omega = 2\pi(N-1)/N$.

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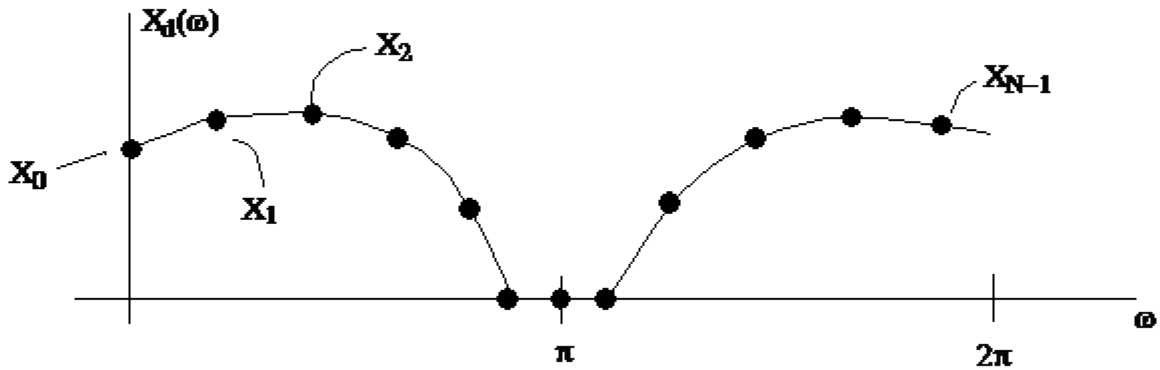

Figure 2.20: The DFT as samples of the DTFT, i.e $X[k] = X_d(\frac{2\pi k}{N})$.

2.5.1.2 Linearity

The DFT can be viewed as a linear operation as follows

$$x[n] \overset{DFT}{\longleftrightarrow} X[k], y[n] \overset{DFT}{\longleftrightarrow} Y[k] \implies z[n] = ax[n] + by[n] \overset{DFT}{\longleftrightarrow} aX[k] + bY[k].$$

This result can be readily shown by substituting $z[n] = ax[n] + by[n]$ into the summation in (2.37) and expanding the summation into the two separate terms, one for $X[k]$ and one for $Y[k]$.

2.5.1.3 Circular Time Shift

When a periodic signal $\tilde{x}[n] = \sum_{k=-\infty}^{\infty} x[n + kN]$ is shifted in time, the effect on the DTFS of the resulting periodic signal $\tilde{x}[n - n_0]$ can be viewed as applying a linear phase term $e^{-j2\pi k n_0/N}$ to the original DTFS coefficients. For a finite-length signal $x[n]$, we cannot define a time-shift in the same manner, by simply shifting the time index, as this would require evaluating the signal $x[n]$ outside the range over which it is defined, namely $0 \leq n \leq N - 1$. However, we can use the time shift property of the DTFS to relate a slightly different notion for finite length signals. By considering a time shift of n_0 samples of the underlying infinite-length periodic signal $\tilde{x}[n - n_0]$ as a "circular shift" of the finite-length sequence $x[n]$, we can study apply the time-shifting property of the DTFS to finite-length signals. To this end, we define a circular shift of a finite-length sequence, $x[n]$ as follows. First, we define the modulo operator, $\ll k \gg_N = k \bmod N$, as,

$$\ll k \gg_N = r, \text{ where } k = \ell N + r,$$

for any integer ℓ. For example $\ll 4 \gg_7 = 7, \ll 7 \gg_4 = 3, \ll 4 \gg_4 = 0, \ll -5 \gg_4 = 3, \ll -2 \gg_4 = 2$. Note that it is sometimes helpful to use $\ll -k \gg_N = N - \ll k \gg_N$. Now we can define the finite-length signal $y[n]$ as a circular shift of the finite-length sequence $x[n]$ as

$$y[n] = \tilde{x}[n - n_0] = x[\ll n - n_0 \gg_N],$$

i.e. a circular shift of the finite-length signal $x[n]$ by n_0 samples, is written $x[\ll n - n_0 \gg_N]$ but can be viewed as the result of taking a single period of the periodic signal $\tilde{x}[n - n_0]$, over the range $0 \leq n \leq N - 1$. The corresponding effect on the DFT of the sequence can be compactly represented as follows

$$x[n] \overset{DFT}{\longleftrightarrow} X[k] \implies y[n] = x[\ll n - n_0 \gg_N] \overset{DFT}{\longleftrightarrow} X[k] e^{-j\frac{2\pi k}{N} n_0}.$$

We see that a circular shift in time of a finite length signal corresponds to a modulation in frequency by a phase term that is linear with frequency with a slope that is proportional to the delay. Note that the resulting DFT is exactly the same as that which we would have obtained by first periodically extending the sequence $x[n]$ to the signal $\tilde{x}[n]$, and taking the DTFS representation of the resulting time-shifted signal $\tilde{x}[n - n_0]$.

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2.5.1.4 Frequency Shift

From the analogous property of the DTFS, we can obtain the relation

$$x[n] \overset{DFT}{\longleftrightarrow} X[k] \implies y[n] = x[n]e^{jk_0\omega_0 n} \overset{DFT}{\longleftrightarrow} X[\ll k - k_0 \gg_N],$$

where $\omega_0 = \frac{2\pi}{N}$, and where we have used the modulo notation to enable the resulting DFT to remain finite-length.

2.5.1.5 Time Reversal

From the DTFS time-reversal property, we obtain,

$$x[n] \overset{DFT}{\longleftrightarrow} X[k] \implies y[n] = x[\ll -n \gg_N] = x[\ll N - n \gg_N] \overset{DFT}{\longleftrightarrow} X[\ll -k \gg_N] = X[\ll N - k \gg_N],$$

where the modulo operator in $\ll N - n \gg_N$ and $\ll N - k \gg_N$ only comes into use for $n = 0$ and $k = 0$ since for all other values, the terms within the modulo operator are within the range of $0, ...N - 1$. This corresponds in the DFT representation to changing the sign of the DFT frequency index, where, to keep the terms within the range from 0 to N, we add N to the index, and take the result modulo N, which results in a reversal of the order of the DFT coefficients $X[\ll N - k \gg_N] = X[N- \ll k \gg_N]$.

2.5.1.6 Conjugate Symmetry

The effect of conjugating a complex-valued signal on its DFT representation can be seen from the DTFS property as

$$x[n] \overset{DFT}{\longleftrightarrow} X[k] \implies x^*[n] \overset{DFT}{\longleftrightarrow} X^*[\ll N - k \gg_N].$$

When the periodic signal $x[n]$ is real valued, i.e. $x[n]$ only takes on values that are real numbers, then the DFT exhibits a symmetry property. This arises directly from the definintion of the DFT, and that real numbers equal their conjugates, i.e. $x[n] = x^*[n]$, such that

$$x[n] = x^*[n] \overset{DFT}{\longleftrightarrow} X[k] \implies X[k] = X^*[\ll N - k \gg_N].$$

2.5.1.7 Products of Signals

When two finite-length signals of the same length are multiplied in time, such that $z[n] = x[n]y[n]$, the resulting signal remains finite length with the same length by definintion. Hence, each of the three signals admit DFT representations using the same set of harmonically related signals. We can observe the effect on the resulting DFT from the analogous DTFS property,

$$x[n] \overset{DFT}{\longleftrightarrow} X[k], y[n] \overset{DFT}{\longleftrightarrow} Y[k] \implies z[n] = x[n]y[n] \overset{DFT}{\longleftrightarrow} \frac{1}{N}\sum_{m=0}^{N-1} X[m]Y[\ll k - m \gg_N].$$

2.5.1.8 Circular Convolution

A dual relationship to that of multiplication in time, is multiplication of DFT coefficients. Specifically, when the two signals $x[n]$ and $y[n]$ are each of finite length N, the finite length signal $z[n]$ of length N, whose DFT representation is given by $Z[k] = X[k]Y[k]$ corresponds to a circular convolution of the signals $x[n]$ and $y[n]$. This leads to the following property of the DFT,

$$x[n] \overset{DFT}{\longleftrightarrow} X[k], y[n] \overset{DFT}{\longleftrightarrow} Y[k] \implies z[n] = \sum_{m=0}^{N-1} x[m]y[\ll n - m \gg_N] \overset{DFT}{\longleftrightarrow} Z[k] = X[k]Y[k].$$

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Length-N Discrete Time Signal	Discrete Fourier Transform
$x[n], 0 \leq n \leq N-1$	$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, 0 \leq k \leq N-1$
$ax[n] + by[n]$	$aX[k] + bY[k]$
$x[\ll n-d \gg_N]$	$X[k]e^{-j2\pi kd/N}$
$x[n]e^{j2\pi \ell n/N}$	$X[\ll k-\ell \gg_N]$
$x[\ll -n \gg_N]$	$X[\ll N-k \gg_N]$
$x^*[n]$	$X^*[\ll N-k \gg_N]$
$x[n]y[n]$	$\frac{1}{N}\sum_{m=0}^{N-1} X[m]Y[\ll k-m \gg_N]$
$\sum_{m=0}^{N-1} x[m]y[\ll n-m \gg_N]$	$X[k]Y[k]$
Parseval's relation: $\sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \frac{1}{N}\sum_{k=0}^{N-1} \lvert X[k]\rvert^2$.	
$x[n]$ real-valued	$X[k] = X^*[\ll N-k \gg_N]$
$x[n]$ real-valued	$\Re\{X[k]\} = \Re\{X[\ll N-k \gg_N]\}$
$x[n]$ real-valued	$\Im\{X[k]\} = -\Im\{X[\ll N-k \gg_N]\}$
$x[n]$ real-valued	$\lvert X[k]\rvert = \lvert X[\ll N-k \gg_N]\rvert$
$x[n]$ real-valued	$\angle X[k] = -\angle X[\ll N-k \gg_N]$

Table 2.5: Discrete Fourier Transform Properties

2.5.1.9 Parseval's relation

The energy containted within the finite duration of a finite length signal can also be computed in terms of its DFT representation using Parseval's relation,

$$x[n] \overset{DFT}{\longleftrightarrow} X[k] \implies \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2.$$

The properties of the DFT discussed in this section are summarized in Table (2.5).

2.6 Applications of spectral representations and signal analysis, DFT spectral analysis

In Section2.5 the DFT was introduced as a method for representing finite-length signals using a linear combination of simpler, finite-length segments of complex exponential signals, in a manner analogous to the construction of periodic signals from the periodic extentions of the same complex exponential family of signals. In Chapter 8 we will explore a number of fast algorithms for explicitly computing the DFT. One of the primary drivers of tremendous growth in the application and use of digital signal processing was, in fact, this discovery. There are two important aspects of the DFT that make its use in digital signal processing systems so pervasive. First, because the DFT is represented by a finite-length sum over quantities that are available, namely the signal $x[n]$,whose DFT is desired, and complex numbers of the form $e^{j2\pi kn/N}$,which can be easily represented using a pair of real numbers in the digital signal processor, one of the real part and one for the imaginary part, the DFT can be exactly computed. This is in contrast to the CTFT and the DTFT which contain either infinite-length integrals or infinite-length summations that cannot be exactly computed. Second, and perhaps more importantly, as we will see in Chapter 8, the DFT can be computed so efficiently that we can make use of many of the DFT properties developed in this and later chapters for performing a wide variety of operations on the signals of interest to our application. For example, we will see how linear filtering can be very efficiently accomplished in the frequency domain through the multiplication of the DFT representation of signals. We will also explore how this can be accomplished for infinite-length signals by stitching together the outputs generated through the use of the finite-length representation generated through the DFT.

Another equally desireable application that can be well-handled using the DFT is the process of spectral analysis. In many applications, including radio communications, image restoration, video compression, even

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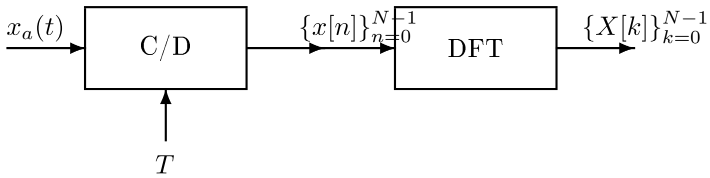

Figure 2.21: System for computing approximate samples of the CTFT of the analog signal $x_a(t)$.

analog circuit analysis, it is important to have a direct measure of the frequency content of signals of interest. This can be accomplished in the laboratory using equipment known as a spectrum analyzer, in which the signal of interest is decomposed into its Fourier components, in real-time, and an image of the frequency composition of the signal is displayed on a screen or an oscilloscope. Such measures are extremely useful and require real-time processing of analog signals in order to privide an accurate picture of the frequency composition of a signal as either the signal or its environment is changing. While for analog signals, it is the continuous-time Fourier transform that provides information about the frequency composition of the signal, as we will explore in more detail in Chapter 4, many of the frequency-domain properties of signals and systems can accurately be measured by discrete-time processing of sampled versions of analog signals.

To make the process more concrete, suppose that we are interested in the frequency content of an analog signal $x_a(t)$, that is known to be approximately bandelimited to B Hz. That is, suppose that we know that

$$x_a(t) \overset{CTFT}{\longleftrightarrow} X_a(\Omega) \approx 0, |\Omega| > 2\pi B$$

i.e., that the CTFT of the signal $x_a(t)$, given by $X_a(\Omega)$ is known to be approximately zero outside of the frequency range $|\Omega| < 2\pi B$. However, we don't know the precise frequency composition of $x_a(t)$, and would like to obtain a real-time measurement of $X_a(\Omega)$ based on processing $x_a(t)$. One way to accomplish this is using DFT. Recall that the DFT of the finite-length signal $x[n]$ enables us to compute samples of the DTFT of the infinite-length zero-padded signal $x_{zf}[n]$. We also know from Chapter 3and will explore in greater detail in Chapter 4, how the CTFT of an analog signal $x_a(t)$ is related to the DTFT of a sampled-version of that signal, $x[n] = x_a(nT)$. We will now explore how these notions can be put together to provide a means for constructing a real-time spetrum analyzer using the DFT.

We will approach this problem as follows. Given access to the analog signal $x_a(t)$,we want to compute approximate samples of $X_a(\Omega)$. Here is the proposed method for accomplishing this task. We will assume that the signal of interest $x_a(t)$ has finite support on the interval $[0,(N-1)T]$ and is nearly bandlimited as described above. That is, we are interested in the signal in the range $t \in [0,(N-1)T]$, where T is a parameter of our sampling process that we will be able to control. The signal $x_a(t)$ to be processed by the processor depicted in Figure 2.21to yeild our samples,

is sampled at a rate of one sample every T seconds producing the discrete-time signals $x[n] = x_a(nT)$. Given the duration of interest, we obtain N samples, which we compactly represent as $\left\{x[n]\right\}_{n=0}^{N-1}$. These samples are then processed with the DFT analysis equation (2.37) producing the values $\left\{X[k]\right\}_{k=0}^{N-1}$. In the frequency domain we know that the following relationship between the continuous-time Fourier transform (CTFT) of $x_a(t)$ and the discrete-time Fourier transform (DTFT) of $x[n]$ holds,

$$X_d(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(\frac{\omega + 2k\pi}{T}\right).$$

That is, if $x_a(t)$ is nearly bandlimited and T is small enough, there is little detrimental aliasing so that then $X_d(\omega)$ will contain copies of $X_a(\Omega)$ scaled to fit within the digital frequency range $|\omega| < \pi$ and scaled in amplitude by a factor of $1/T$. This is depicted in Figure (2.22).

The DFT, $\{X[k]\}_{k=0}^{N-1}$ is a set of samples of $X_d(\omega)$ on the interval $[0, 2\pi]$ such that $X[k] = X_d(\frac{2\pi k}{N})$, $0 \le k \le N-1$. Thus, we have (for N odd):

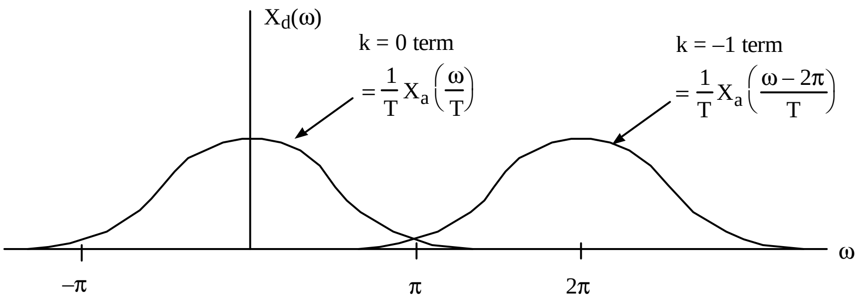

Figure 2.22: Spectrum of $x_a(t)$ scaled in amplitude and frequency and replicated on the digital frequency axis according to the relation $X_d(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(\frac{\omega+2k\pi}{T}\right)$.

$$X[k] \approx \begin{cases} \frac{1}{T}X_a\left(\frac{2\pi k}{NT}\right), & 0 \le k \le \frac{N-1}{2} \\ \frac{1}{T}X_a\left(\frac{2\pi(k-N)}{NT}\right), & \frac{N-1}{2} < k \le N-1. \end{cases}$$

So, we have that $\{X[k]\}_{k=0}^{N-1}$ actually contain approximate, scaled samples of $X_a(\Omega)$. It is important to note that the first half of the DFT samples provide samples of $X_a(\Omega)$ for $\Omega > 0$, while the second half of the DFT gives samples of $X_a(\Omega)$ for $\Omega < 0$. This peculiarity arises due to the definition of the DFT, whereby $\{X[k]\}_{k=0}^{N-1}$ is a set of samples of $X_d(\omega)$ on the interval $[0, 2\pi]$ rather than on $[-\pi, \pi]$.

2.6.1 DFT Spectral Analysis of Sinusoids

Suppose that we wish to determine the frequency content of an analog signal that containes a number of sinusoidal components, of the form

$$x_a(t) = \sum_{i=1}^{M} A_i \cos(\Omega_i t),$$

and we have available only the following samples of the signal, $x[n] = x_a(nT)$, for $0 \le n \le N-1$. From this set of samples, $\{x[n]\}_{n=0}^{N-1}$, we wish to determine as much information about $x_a(t)$ as possible. Namely, we would like to determine the values of M, $\{\Omega\}_{i=1}^{M}$and $\{A_i\}_{i=1}^{M}$.

One approach to this problem is to use DFT spectral analysis as depicted in Figure 2.21. Variations on this type of problem arise in numerous applications, including such wide-ranging topics as digital communications, radio astronomy, and, in particular, applications involving rotating machinery. For example, an acoustic transducer coupled to a piece of rotating machinery will output a periodic signal (sum of sinusoids) plus, perhaps, a smaller nonperiodic component. A frequency (i.e., DFT) analysis of this signal can indicate whether the machinery requires maintenance or replacement. Similarly, in an underwater setting, ships can be identified through DFT analysis of the acoustic signals they emit, which are collected by hydrophones. Consider a single sinusoidal component, captured within $x[n]$ as

$$x[n] = A\cos(\Omega_0 nT),$$

for $0 \le n \le N-1$.The DFT of $x[n]$ is the set of samples of the DTFT:

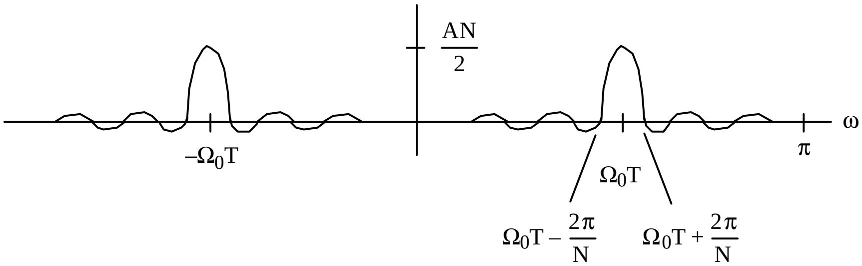

Figure 2.23: Schematic depiction of two periodic sinc functions at $\omega = \pm\Omega_0 T$.

$$
\begin{aligned}
X(\omega) &= \sum_{n=0}^{N-1} A\cos(\Omega_0 nT)e^{-j\omega n}, \\
&= \sum_{n=0}^{N-1} \frac{A}{2}\left(e^{-j(\omega-\Omega_0 T)n} + e^{-j(\omega+\Omega_0 T)n}\right) \\
&= \underbrace{e^{-j(\omega-\Omega_0 T)\frac{N-1}{2}} \frac{\frac{A}{2}\sin\left((\omega-\Omega_0 T)\frac{N}{2}\right)}{\sin\left((\omega-\Omega_0 T)\frac{1}{2}\right)}}_{T_1(\omega)} + \underbrace{e^{-j(\omega+\Omega_0 T)\frac{N-1}{2}} \frac{\frac{A}{2}\sin\left((\omega+\Omega_0 T)\frac{N}{2}\right)}{\sin\left((\omega+\Omega_0 T)\frac{1}{2}\right)}}_{T_2(\omega)}.
\end{aligned}
$$

The two periodic sincs $T_1(\omega)$ and $T_2(\omega)$ above have peaks at $\omega = \pm\Omega_0 T$ and give rise to a DTFT that appears as in Figure 2.23.

The width of each main lobe is $\frac{4\pi}{N}$ based on the zero crossings of the periodic sinc function. If N is large enough, the main contributions from these pulses do not substantially overlap in frequency much and we obtain that

$$
\begin{aligned}
|X_d(\omega)| &= |T_1(\omega) + T_2(\omega)| \\
&\approx |T_1(\omega)| + |T_2(\omega)| \\
&= \frac{A}{2}\left|\frac{\sin\left((\omega-\Omega_0 T)\frac{N}{2}\right)}{\sin\left((\omega-\Omega_0 T)\frac{1}{2}\right)}\right| + \frac{A}{2}\left|\frac{\sin\left((\omega+\Omega_0 T)\frac{N}{2}\right)}{\sin\left((\omega+\Omega_0 T)\frac{1}{2}\right)}\right|.
\end{aligned}
$$

Therefore, $\{X[k]\}_{k=0}^{N-1}$ will provide approximate samples of the magnitude of the above plot,

$$
\begin{aligned}
|X[k]| &= \left|X_d\left(\frac{2\pi k}{N}\right)\right| \\
&\approx \frac{A}{2}\left|\frac{\sin\left((\frac{2\pi k}{N}-\Omega_0 T)\frac{N}{2}\right)}{\sin\left((\frac{2\pi k}{N}-\Omega_0 T)\frac{1}{2}\right)}\right| + \frac{A}{2}\left|\frac{\sin\left((\frac{2\pi k}{N}+\Omega_0 T)\frac{N}{2}\right)}{\sin\left((\frac{2\pi k}{N}+\Omega_0 T)\frac{1}{2}\right)}\right|,
\end{aligned}
$$

from which we can estimate Ω_0 and A. If $x[n]$ is obtained from samples of a signal containing a sum of several sinusoids, we will have multiple peaks and we can estimate all of the parameters M, Ω_i, and $A_i$using one of a number of possible methods, depending on what other information we have about the environment in which the signals have been obtained. For example, we might use the following algorithm:

$$
\begin{aligned}
M &= \text{Number of peaks detected on the interval } [0,\pi] \\
\Omega_i T &= \text{Location of the } i-th \text{ peak} \\
\frac{NA_i}{2} &= \text{Amplitude of the } i-th \text{ peak}
\end{aligned}
$$

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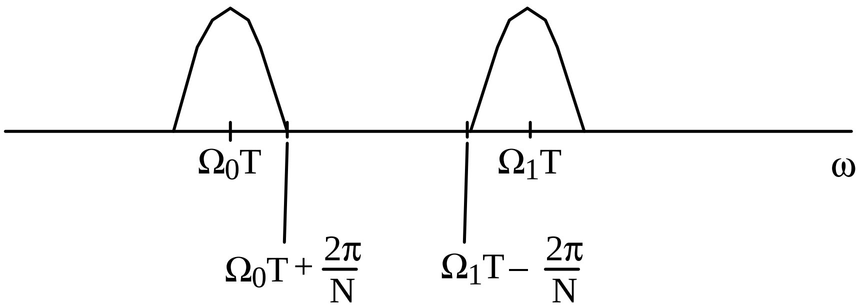

Figure 2.24: Two closely-spaced peaks in the spectral analysis of a signal containing sinusoidal components.

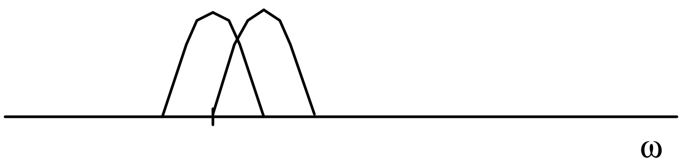

Figure 2.25: Two closely spaced peaks that overlap by 50% of their maximum amplitude.

Actually, sinusoidal spectral analysis, via the DFT, is not usually this straightforward. Some of the problems that frequently arise include:

1. Two Ω_i are so close together that the two peaks add together into a single peak, or do not easly resolve into two distinct peaks. One possible solution is to use a larger value of N, i.e., collect data over a longer observation interval so that peaks will be narrower and will not overlap. Suppose that the signal contains the two frequencies Ω_0 and Ω_1. Then the main lobes of the periodic sincs may look like that shown in Figure2.24. To clearly distinguish the two peaks, we might require that the main lobes not overlap, i.e.,

$$\begin{aligned}\Omega_1 T - \frac{2\pi}{N} &< \Omega_0 T + \frac{2\pi}{N}\\ (\Omega_1 - \Omega_0)\,T &> \frac{4\pi}{N}\\ NT &> \frac{4\pi}{(\Omega_1 - \Omega_0)}.\end{aligned}$$

   Therefore, the necessary observation interval NT increases as the frequency separation $(\Omega_1 - \Omega_0)$ decreases. So, the closer two sinusoidal components are to one another, the longer you will need to wait, in order to collect sufficient data to resolve the two peaks. In practice the above condition on NT is often too conservative, since we can still discern two peaks even if there is some overlap. If we instead require that there be no more than 50% overlap as shown in Figure 2.25, then the required condition becomes m

$$\begin{aligned}\Omega_1 T - \frac{2\pi}{N} &> \Omega_0 T\\ NT &> \frac{2\pi}{(\Omega_1 - \Omega_0)}.\end{aligned}$$

2. In the previous analysis, we ignored the effect of the amplitude of the two frequency components, $A_0$and A_1. If two of the frequencies, Ω_i are fairly close together and one of the A_i is much smaller than the other, this may cause the small peak to be buried in the sidelobes of the larger peak. One possible solution to this problem is to consider applying a window to the data prior to computing the DFT, to reduce the resulting sidelobes. Since the DTFT of the signal $x[n] = w[n]x_a(nT)$ implicitly contains the window function $w[n]$ multiplying the signal $x_a(nT)$, then the effect of the wundown $w[n]$ on the

resulting DTFT, and therefore the resulting DFT, must be considered. In the analysis so far, we have assumed that a rectangular window, $w[n] = 1$,for $0 \leq n \leq N-1$,was used. By selecting a window $w[n]$ that has lower side-lobe behavior than the rectangular window, which has a periodic sinc function as its DTFT, we may be able to aleviate some of the side lobe issues. This is a matter that is discussed in more detail in chapter 6.Unfortunately, however, this is not a cure-all. While windowing reduces the size of the resulting side lobes near the peak, it also has the effect of widening the main lobe so that closely spaced sinusoids will be harder to distinguish.

3. The most important difficulty in spectral analysis is the presence of noise in the signal aquisition environment and in the signal itself. There are a host of methods that could have been used for sinusoidal spectral analysis that would work perfectly (i.e. perfect estimation of all parameters) if there were no noise. Properly handling the spectral estimation problem in the presence of noise is a topic of great interest in the digital spectral analysis field and a number of methods exist, whose particular implementation and efficacy depends strongly on the type of noise that is present and on the signal to noise ratio, i.e. the relative strength of the signal components $A_i$and the noise power in the frequency range of interest. The achievable resolution in spectral analysis ultimately depends on properties of the noise, and on NT, the length of observation interval of $x_a(t)$.

2.6.2 Zero-Padding in DFT Spectral Analysis

Consider a spectral estimation problem as described in the previous section, in which we have obtained a set of samples $\{x[n]\}_{n=0}^{N-1}$, from an analog signal, such that $x[n] = x_a(nT)$,for $0 \leq n \leq N-1$. Let $x_{zp}[n]$ be the infinite length signal obtained by letting $x_{zp}[n] = x[n]$, for $0 \leq n \leq N-1$, and set to zero outside of this interval. We then have that the DFT of $x[n]$is related to the DTFT of $x_{zp}[n]$ through

$$X[k] = X_d\left(\frac{2\pi k}{N}\right), 0 \leq k \leq N-1.$$

We now consider another finite-length signal $x_2[n]$, of length $N+M$, such that

$$x_2[n] = \begin{cases} x[n], & 0 \leq n \leq N-1 \\ 0 & N \leq n \leq N+M-1, \end{cases}$$

so that the resulting finite-length signal contains both the original set of samples $x[n] = x_a(nT)$, as well as the additional M zero values from $x_{zp}[n]$ that follow these samples in the signal $x_{zp}[n]$. Let us consider how the length $N+M$ DFT of the signal $x_2[n]$ relates to the DTFT of the signal$x_{zp}[n]$. From the definition of the DFT, we have

$$\begin{aligned} X_2[k] &= \sum_{n=0}^{N+M-1} x_2[n]e^{-j\frac{2\pi k}{N+M}n} \\ &= \sum_{n=0}^{N-1} x_{zp}[n]e^{-j\frac{2\pi k}{N+M}n} \\ &= \left.\sum_{n=0}^{N-1} x_{zp}[n]e^{-j\omega n}\right|_{\omega=\frac{2\pi k}{N+M}} \\ &= X_d\left(\frac{2\pi k}{N+M}\right), \end{aligned}$$

i.e., $X_2[k]$ provides a set of $N+M$ samples of the DTFT of $x_{zp}[n]$ equally spaced over the interval $\omega \in [0, 2\pi]$. That is, zero padding of the signal prior to taking the DFT provides a more densely spaced set of samples of precisely the same DTFT as the original, non-zero padded length N signal, $x[n]$. One reason that zero-padding is often used in DFT spectral analysis is the desire for a more densely spaced set of samples of $X_d(\omega)$, for example, to make a visual plot of the samples as an approximation to the true $X_d(\omega)$, which is a function of the continuous variable, ω. A second, and often more compelling reason for zero padding,

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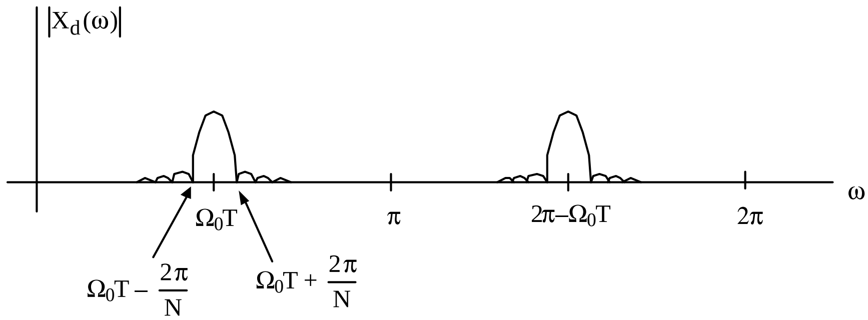

Figure 2.26: DTFT Magnitude for the sequence $x[n] = \cos(\Omega_0 nT)$.

arises when the sequence length, N, is not a length for which a fast DFT implementation is available. For example, we will explore a number of efficient implementations of the DFT, known collectively as the fast Fourier transform, or FFT, algorithms in chapter 8, for which N must be a power of two, i.e., $N = 2^\nu$. As a result, it may be desireable to pad the length of the sequence out to the nearest length for which an FFT implementation is available. This is a consideration of particular importance if N is large, or if many length-N DFTs are needed.

Example: DFT Spectral Analysis of Sinusoids

If $x_a(t) = \cos(\Omega_0 t)$, and we have available the set of samples, $\{x[n]\}_{n=0}^{N-1}$, where $x[n] = \cos(\Omega_0 nT)$, then the magnitude of the

DTFT of this sequence looks as shown in Figure 2.26.

Computing the DFT of the sequence $\{x[n]\}_{n=0}^{N-1}$ provides samples of $X_d(\omega)$. Depending on the length, N, of the DFT and the value of $\Omega_0 T$, there may or may not be samples of the DFT at or near the peaks of the mainlobes and sidelobes of the DTFT. Zero-padding prior to the DFT will yield a denser collection of samples of $X_d(\omega)$ and will therefore provide a better representation of $X_d(\omega)$, especially when plotted graphically.

The use of one of a variety of suitably designed windows, $\{w[n]\}_{n=0}^{N-1}$, can aid in DFT spectral analysis when the window is applied to the data prior to taking the DFT, i.e. $Y[k] = \sum_{n=0}^{N-1} w[n]x[n]e^{-j2\pi kn/N}$. Note the sequence $\{x[n]\}_{n=0}^{N-1}$, already implicitly contains a rectangular window, $w[n] = 1$, $0 \leq n \leq N-1$. A different window could be applied by simply computing $Y[k] = \sum_{n=0}^{N-1} w[n]x[n]e^{-j2\pi kn/N}$, for a suitable window sequence, such as Hamming window, prior to computing the DFT. This will widen main lobe (by a factor of two) but will also greatly reduce the sidelobe behavior.

Short Time Fourier Transform (or Windowed Fourier Transform)

In practice, for long signals, such as speech or radio transmissions, it is informative to examine the frequency content of the signal changes over time. To enable this, we can parse the signal into shorter blocks of time, which are often overlapped with one-another, and compute the DFT of each of these blocks. This process is often termed the short-time Fourier transform (STFT), and a graphical display of the magnitude of the STFT is called the spectrogram of the signal. Figure 2.27illustrates the process in computing the STFT.

When the signal of interest is a speech signal, as shown in Figure2.28, the STFT of the signal is often shown as an image, as it appears in the figure, right.

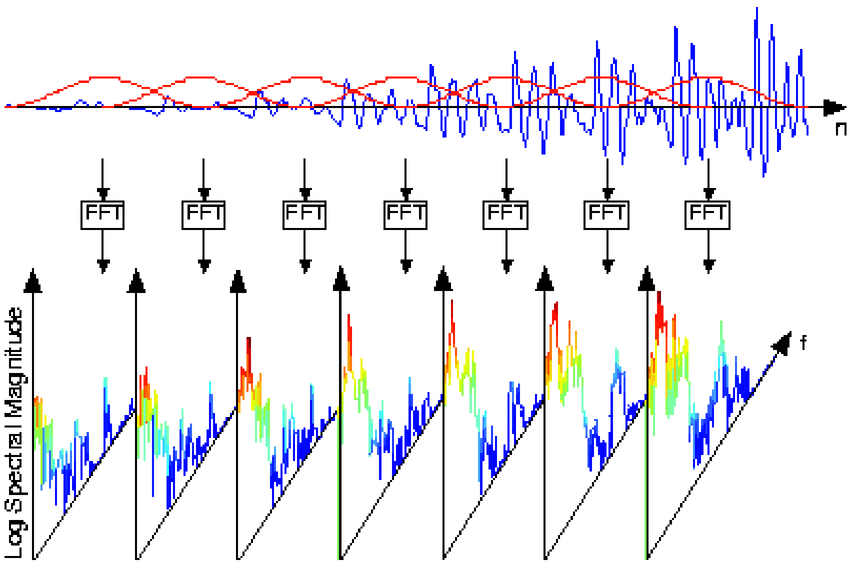

Figure 2.27: Graphical depiction of the process of computing the short-time Fourier transform of a long signal by applying a sequence of overlapping windows. These windowed segments of the longer signal are then processed using the fast Fourier transform (FFT) algorithm for computing the DFT, whose log magnitude is then shown as a function of time.

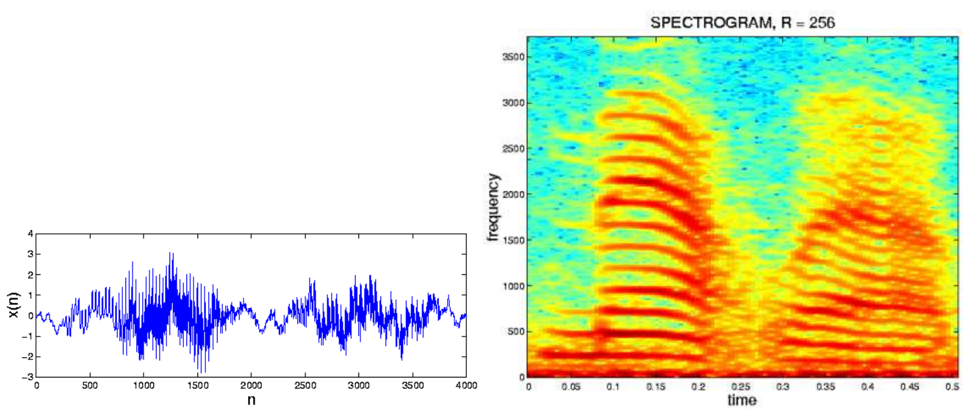

Figure 2.28: A segment of a signal generated from samples of a speech signal and its spectrogram computed using the code shown in table 2.6.

```
% LOAD DATA
load mtlb;
x = mtlb; % x holds the speech signal
figure(1), clf
plot(0:4000,x)
xlabel('n')
ylabel('x(n)')
% SET PARAMETERS
R = 256; % R: block length
window = hamming(R); % window function of length R
N = 512; % N: frequency discretization
L = 35; % L: time lapse between blocks
fs = 7418; % fs: sampling frequency
overlap = R - L;
% COMPUTE SPECTROGRAM
[B,f,t] = specgram(x,N,fs,window,overlap);
% MAKE PLOT
figure(2), clf
imagesc(t,f,log10(abs(B)));
colormap('jet')
axis xy
xlabel('time')
ylabel('frequency')
title('SPECTROGRAM, R = 256')
```

The key command

```
[B,f,t] = specgram(x,N,fs,window,overlap);
```

computes the short time Fourier transform of a signal using a sliding window. The description of its input and output are as follows.

Input:

x - a vector that holds the input signal

N - specifies the FFT (or DFT) length that specgram uses

fs - a scalar that specifies the sampling frequency

window - specifies a windowing function

overlap - number of samples by which the sections overlap

Output:

B - matrix contains the spectrogram which is the magnitude of the STFT of the input signal

f - a column vector contains the frequencies at which specgram computes the discrete-time Fourier transform. The length of f is equal to the number of rows of B.

t - a column vector of scaled times, with length equal to the number of columns of B. t(j) is the earliest time at which the j-th window intersects x. t(1) is always equal to 0.

Table 2.6: M-file code used to generate the spectrogram shown in Figure 2.28.