ECE 410 **DIGITAL SIGNAL PROCESSING** **D. Munson**
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<u>Frequency Response of Linear Shift Invariant (LSI) Systems</u>

We know that LSI systems have input and output sequences, $x[n]$ and $y[n]$, respectively, that satisfy the convolution sum, that is,

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m],$$

where $h[n]$ is the impulse response of the LSI system. Taking the DTFT of both sides, we have that,

$$\begin{aligned} Y_d(\omega) &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h[m]x[n-m]e^{-j\omega n} \\ &= \sum_{m=-\infty}^{\infty} h[m]\left(\sum_{n=-\infty}^{\infty} x[n-m]e^{-j\omega n} \right) \\ &= \left(\sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m} \right) X_d(\omega) \\ &= H_d(\omega)X_d(\omega), \end{aligned}$$

where, $H_d(\omega)$ is the DTFT of the impulse response $h[n]$ and is referred to as the *frequency response* of the LSI system. This leads us to the relationship in the frequency domain, between the DTFT of the input sequence and that of the output sequence,

$$Y_d(\omega) = H_d(\omega)X_d(\omega).$$

If input is $x_n = e^{j\omega_o n}$ then

$$y_n = \sum_{m=-\infty}^{\infty} h_m \, e^{j\omega_o(n-m)}$$

$$= e^{j\omega_o n} \sum_{m=-\infty}^{\infty} h_m \, e^{-j\omega_o m}$$

$$= H_d(\omega_o) \, e^{j\omega_o n} \qquad (❑)$$

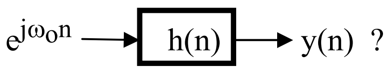
So output is same as input except scaled by the constant $H_d(\omega_o)$,
i.e., ↑ depends on input "frequency"

$$e^{j\omega_o n} \longrightarrow \boxed{h_n} \longrightarrow H_d(\omega_o)\, e^{j\omega_o n}$$

$e^{j\omega_o n}$ is called an "eigen-sequence."

Digression

What does this mean:

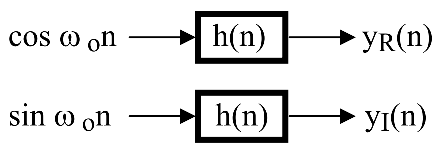


If h(n) is real valued then this means:

$$y(n) = h(n) * [\cos \omega_o n + j \sin \omega_o n]$$

$$= h(n) * \cos \omega_o n + j\, h(n) * \sin \omega_o n$$

i.e., the diagram above is a concise representation of a pair of systems having real-valued inputs and outputs:



with

$$y(n) \triangleq (y_R(n), y_I(n))$$

$$= y_R(n) + j\, y_I(n)$$

Note that we can build this system.

If h(n) is complex-valued, then as we saw in Lecture 18, we can still build system:

Write $h(n) = \underbrace{h_R(n) + j\, h_I(n)}_{\text{real}}$

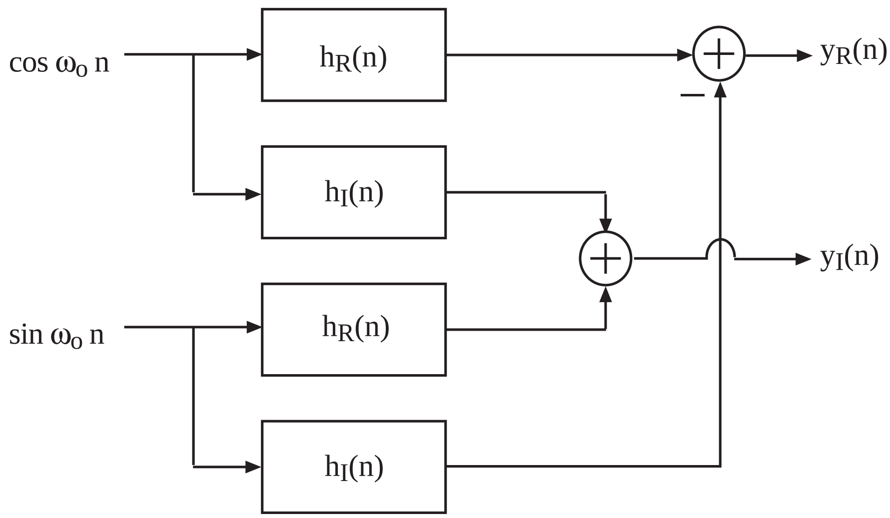
Then

$$y(n) = (h_R(n) + j\, h_I(n)) * (\cos \omega_o n + j \sin \omega_o n)$$

$$= h_R(n) * \cos \omega_o n - h_I(n) * \sin \omega_o n$$

$$+ j\, [h_I(n) * \cos \omega_o n + h_R(n) * \sin \omega_o n]$$

An implementation is:



(End of Digression)

Now, let's go back and use (❑). This equation implies that the response to

$$\cos \omega_o n = \frac{1}{2}\left(e^{j\omega_o n} + e^{-j\omega_o n}\right)$$

is

$$y_n = \frac{1}{2} H_d(\omega_o)\, e^{j\omega_o n} + \frac{1}{2} \underbrace{H_d(-\omega_o)}\, e^{-j\omega_o n}$$

$$\sum_m h_m\, e^{j\omega_o m} \underset{\text{for real } h_m}{=} \left[\sum_m h_m\, e^{-j\omega_o m}\right]^*$$

$$= H_d^*(\omega_o)$$

$$\Rightarrow \quad y_n = \frac{1}{2} \overbrace{\left|H_d(\omega_o)\right| e^{j\angle H_d(\omega_o)}}^{H_d(\omega_o)}\, e^{j\omega_o n}$$

$$+ \frac{1}{2} \overbrace{\left|H_d(\omega_o)\right| e^{-j\angle H_d(\omega_o)}}^{H_d^*(\omega_o)}\, e^{-j\omega_o n}$$

$$= \frac{1}{2}\left|H_d(\omega_o)\right|\left\{e^{j[\omega_o n + \angle H_d(\omega_o)]} + e^{-j[\omega_o n + \angle H_d(\omega_o)]}\right\}$$

$$= \left|H_d(\omega_o)\right| \cos\left(\omega_o n + \angle H_d(\omega_o)\right)$$

Picture:

$$\cos(\omega_o n) \longrightarrow \boxed{h_n} \longrightarrow \left|H_d(\omega_o)\right| \cos\left(\omega_o n + \angle H_d(\omega_o)\right)$$

So, response to $\{\cos\omega_o n\}_{n=-\infty}^{\infty}$ is also a cos with

a) Same frequency
b) Amplitude $\left|H_d(\omega_o)\right|$
c) Phase $\angle H_d(\omega_o)$

<u>Note</u>: This result assumes H(z) is stable, because $H_d(\omega)$ exists only if ROC_H includes the unit circle.

Example

Given $y_n = x_n + 2x_{n-1}$

find the output due to $x_n = \cos\frac{\pi}{2} n \quad \forall \; n.$

<u>Solution</u>

$$Y_d(\omega) = X_d(\omega) + 2\, e^{-j\omega} X_d(\omega)$$

$$\Rightarrow H_d(\omega) = \frac{Y_d(\omega)}{X_d(\omega)} = 1 + 2\, e^{-j\omega}$$

Know $y_n = \left|H_d\left(\frac{\pi}{2}\right)\right| \cos\left(\frac{\pi}{2} n + \angle H_d\left(\frac{\pi}{2}\right)\right)$

Have $H_d\left(\frac{\pi}{2}\right) = 1 + 2\, e^{-j\frac{\pi}{2}} = 1 - j\,2 = \sqrt{5}\; e^{-j63.43^\circ}$

$$\Rightarrow \quad y_n = \sqrt{5}\,\cos\left(\frac{\pi}{2} n - 63.43^\circ\right)$$

Example

Given $H(z) = \dfrac{z}{z - \frac{1}{2}}$

find the output due to $x_n = \cos \frac{\pi}{4} n \quad \forall\ n.$

<u>Solution</u>

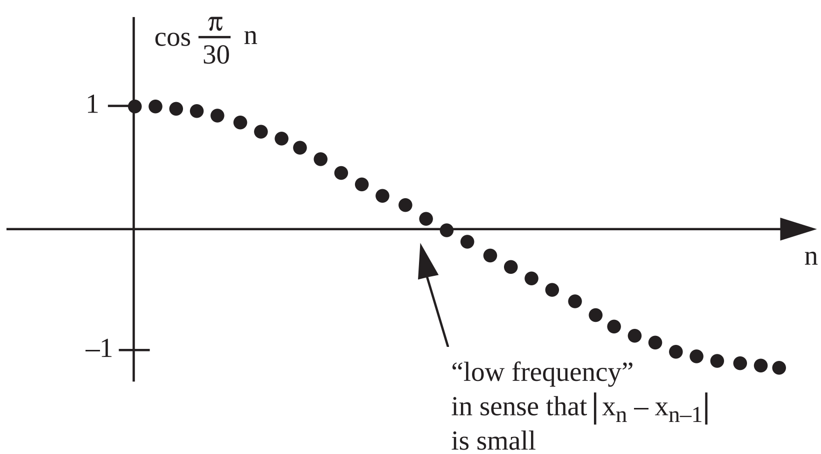
$$H_d(\omega) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

$$\Rightarrow H_d\left(\frac{\pi}{4}\right) = \frac{1}{1 - \frac{1}{2} e^{-j\frac{\pi}{4}}} = \frac{1}{1 - \frac{\sqrt{2}}{4} + j\frac{\sqrt{2}}{4}} = 1.36\, e^{-j28.68^\circ}$$

$$\Rightarrow \quad y_n = 1.36 \cos\left(\frac{\pi}{4} n - 28.68^\circ\right)$$

But, why call ω_o in $\cos(\omega_o n)$ "frequency?"

Suppose $\omega_o = \frac{\pi}{30}$. Plot $x_n = \cos \frac{\pi}{30} n$:



Suppose $\omega_o = \frac{\pi}{2}$. Plot $x_n = \cos \frac{\pi}{2} n$:

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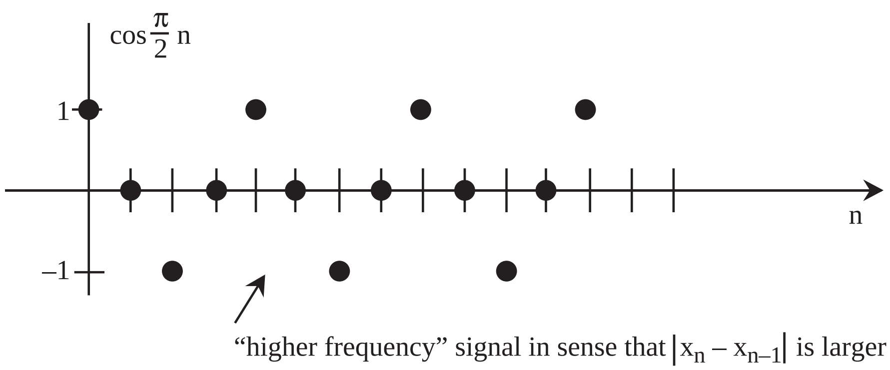


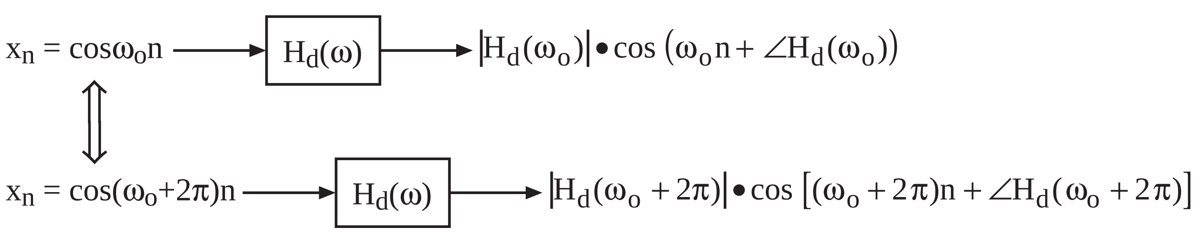
<u>Lowest digital frequency</u>:

$\omega_o = 0 \Rightarrow x_n = \cos(0 \bullet n) = 1 = \text{constant}.$

<u>Highest digital frequency</u>:

$\omega_o = \pi \Rightarrow x_n = \cos(\pi n) = (-1)^n$ so that $|x_n - x_{n-1}|$ is maximized for sinusoid of unit amplitude.

Why is $H_d(\omega)$ periodic with period 2π? <u>Has to be</u>! Note:



But, these two inputs with "different frequencies" are in fact identical sequences. So,

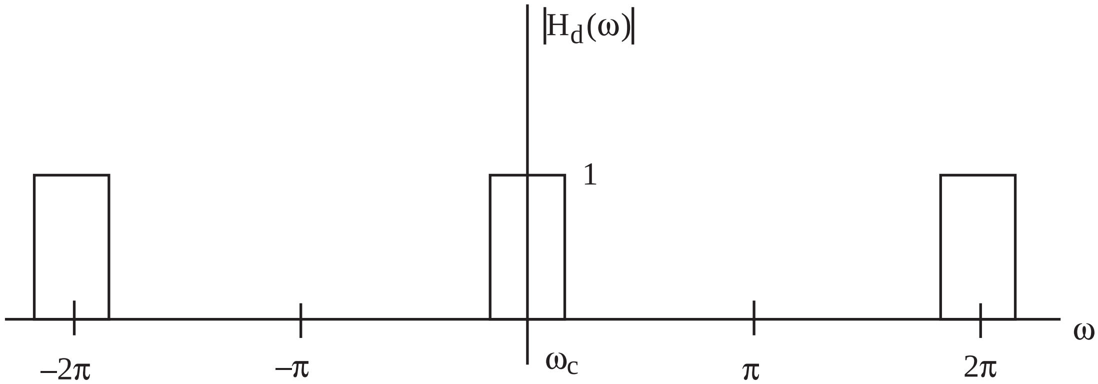
$\Rightarrow \quad |H_d(\omega_o)| \quad = \quad |H_d(\omega_o + 2\pi)|$

and

$\angle H_d(\omega_o) \quad = \quad \angle H_d(\omega_o + 2\pi)$

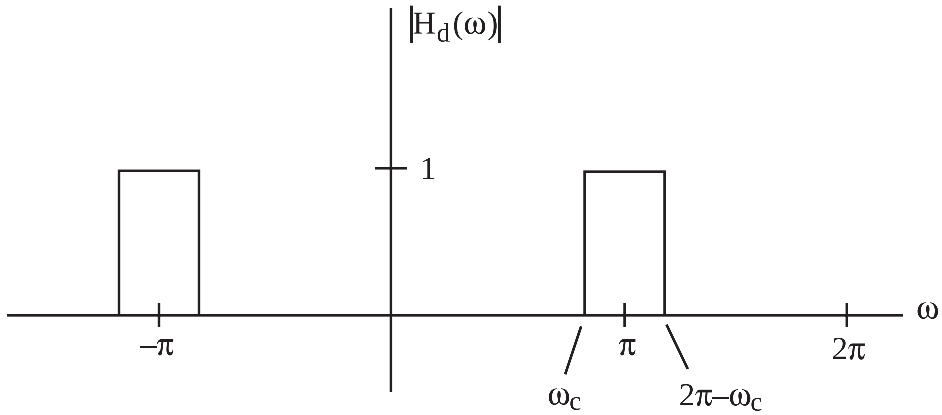
$\Rightarrow \quad H_d(\omega_o) \quad = \quad H_d(\omega_o + 2\pi)$

Ideal digital low-pass filter (LPF) frequency response:



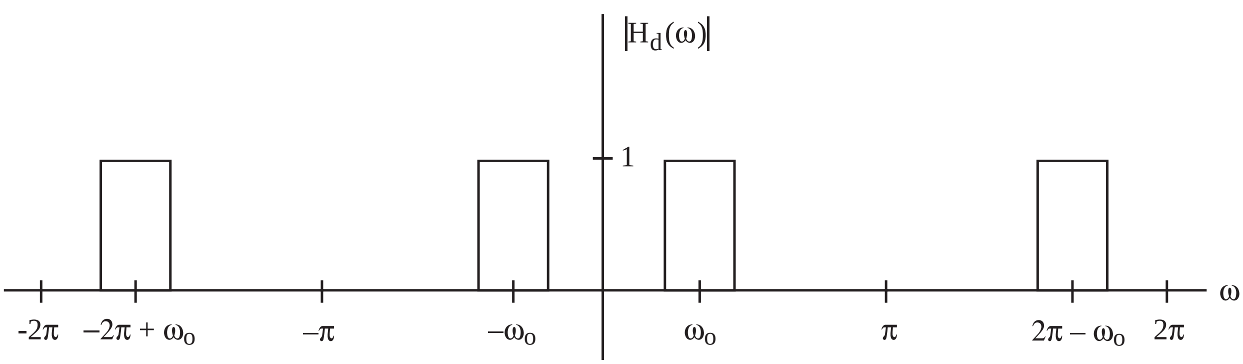
"Passes" all sinusoids having $|\omega_o| \le \omega_c$. Completely attenuates all others.

Ideal digital high-pass filter (HPF) frequency response:



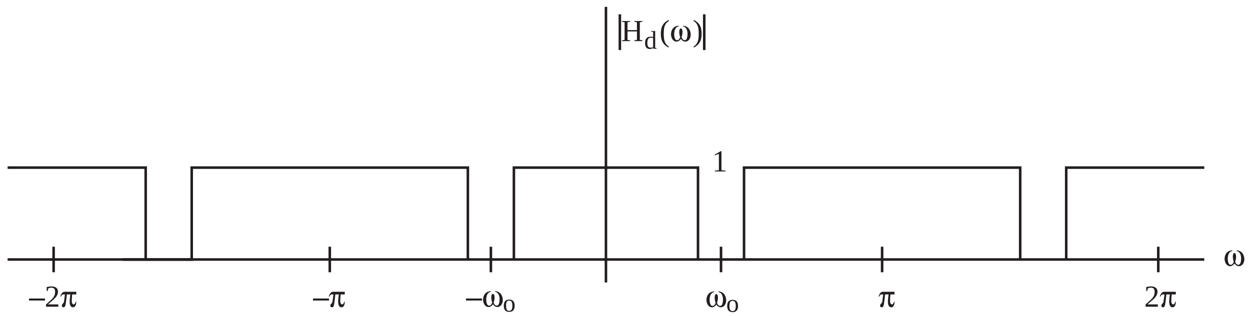
"Passes" all sinusoids having $\omega_c \le |\omega_o| \le 2\pi - \omega_c$. Attenuates all others.

Ideal digital band-pass filter (BPF) frequency response:



Passes all frequencies in band centered at ω_o. Attenuates all others.

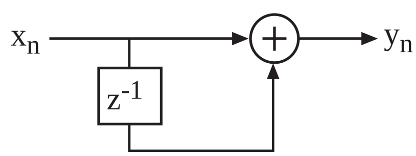
Ideal digital band-stop filter (BSF) frequency response:



Attenuates all frequencies in band centered at ω_0. When the stop-band is narrow, this is also called a notch filter.

Actual frequency responses using finite-order H(z) give only an approximation to ideal $H_d(\omega)$.

Example



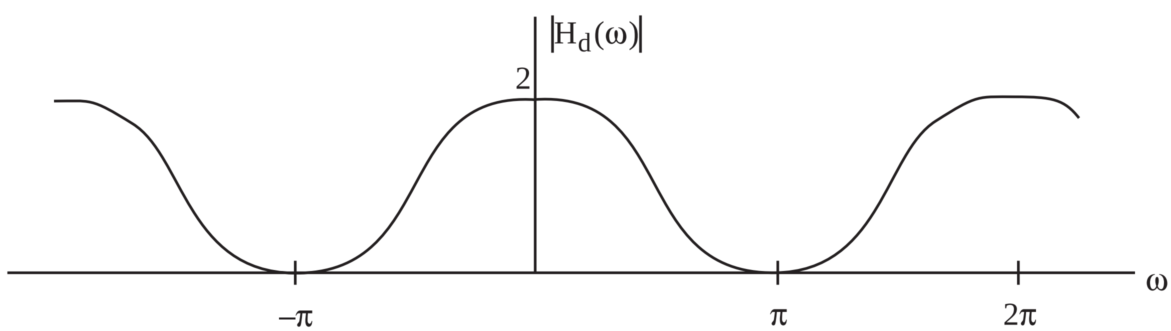
Have:

$$y_n = x_n + x_{n-1}$$

$$\Rightarrow \quad H(z) = 1 + z^{-1}$$

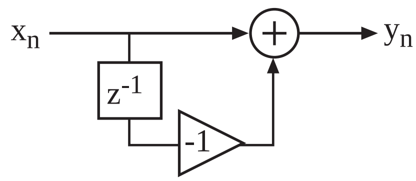
$$\Rightarrow H_d(\omega) = 1 + e^{-j\omega}$$

$$\Rightarrow |H_d(\omega)| = \sqrt{2 + 2\cos\omega}$$



So, this is a crude LPF.

Example



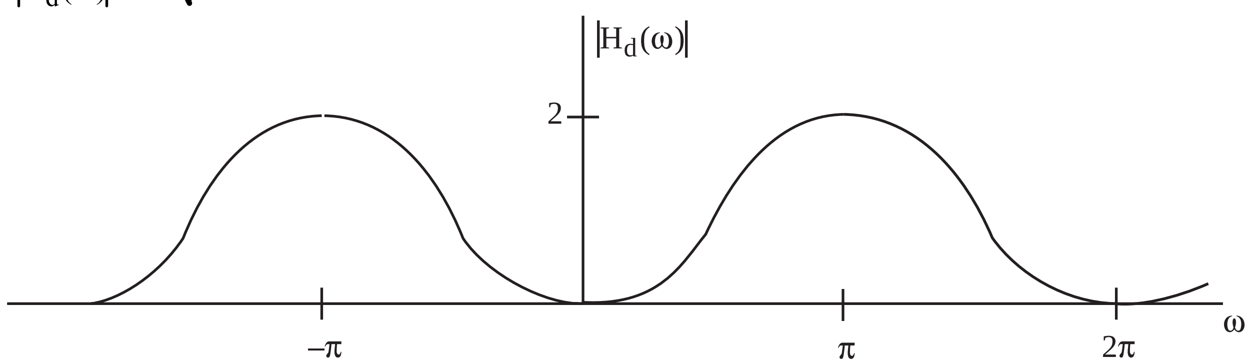
Have:

$$y_n = x_n - x_{n-1}$$

$$\Rightarrow H(z) = 1 - z^{-1}$$

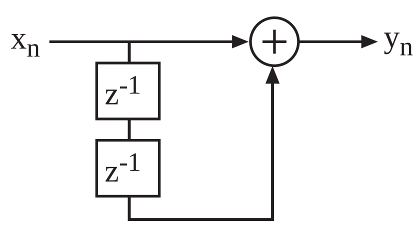
$$\Rightarrow H_d(\omega) = 1 - e^{-j\omega}$$

$$|H_d(\omega)| = \sqrt{2 - 2\cos\omega}$$



So, this is a crude HPF.

Example



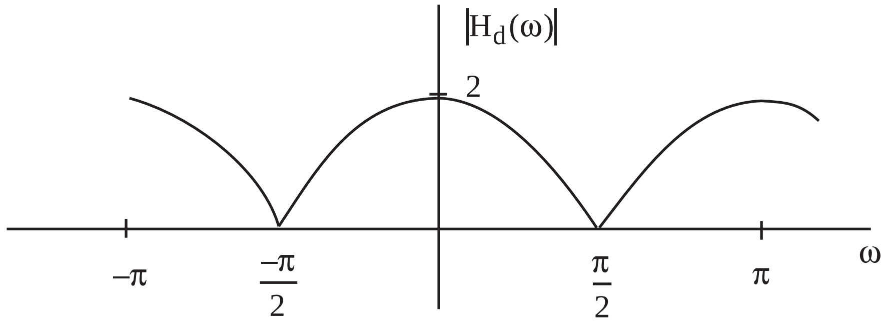
Have:

$$H_d(\omega) = 1 + e^{-j2\omega}$$

$$= e^{-j\omega}\left(e^{j\omega} + e^{-j\omega}\right)$$

$$= e^{-j\omega}\, 2\cos\omega$$

$$\Rightarrow |H_d(\omega)| = 2\,|\cos\omega|$$

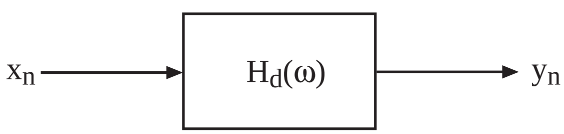


So, this is a crude BSF with stopband centered at $\omega = \frac{\pi}{2}$.

Note: In these last few examples, we have looked at frequency responses of simple nonrecursive filters. We can achieve responses that are much closer to ideal (as close as we would like) by considering nonrecursive filters with more coefficients, and through use of recursive filters. The design of such filters will be an important topic later in the course.

Phase of Frequency Response

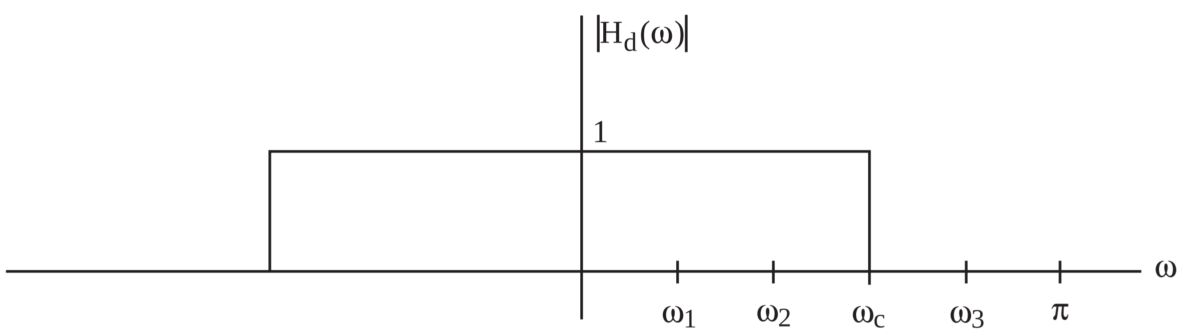
Suppose



with

$$x_n = \cos \omega_1 n + \cos \omega_2 n + \cos \omega_3 n$$

and



Suppose values of ω_1, ω_2 and ω_3 are unknown, but do know $\omega_1, \omega_2 < \omega_c$ and $\omega_3 > \omega_c$. Furthermore, suppose ω_3 is a contaminating sinusoid and you wish to recover just the sinusoids at ω_1 and ω_2.

Thus, want

$$y_n = \cos \omega_1 n + \cos \omega_2 n \qquad (*)$$

How does $\angle H_d(\omega)$ affect y_n?

Know:

$$y_n = \cos\left(\omega_1 n + \angle H_d(\omega_1)\right) + \cos\left(\omega_2 n + \angle H_d(\omega_2)\right)$$

If want (*) then need

$$\angle H_d(\omega) = 0 \text{ for all } \omega \text{ in passband.}$$

$$\Rightarrow H_d(\omega) = |H_d(\omega)|\, e^{j0} = |H_d(\omega)|$$

$$\Rightarrow \quad h_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1\, e^{j\omega n}\, d\omega$$

$$= \frac{\omega_c}{\pi}\,\mathrm{sinc}\,(\omega_c n)$$

↑

noncausal, and large for $n < 0$.

If we wish to design a causal filter, this type of h_n cannot be well approximated.

Suppose we are willing to accept a delayed version of the two lower-frequency sinusoids. That is, suppose instead of (*), we are satisfied with

$$y_n = \cos\left[\omega_1 (n - M)\right] + \cos\left[\omega_2 (n - M)\right]$$

What $\angle H_d(\omega)$ and h_n does this correspond to?

Answer: $\angle H_d(\omega) = -M\omega$ ~ linear phase

Note: It makes sense that we need to shift a higher frequency signal more in phase to get the same delay:

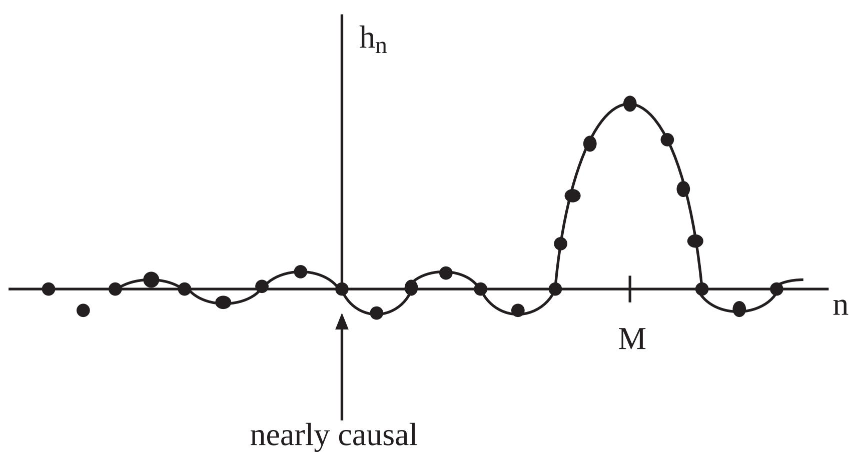
Now, what is h_n for the case with linear phase?

$$h_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1 \bullet e^{-j\omega M}\, e^{j\omega n}\, d\omega$$

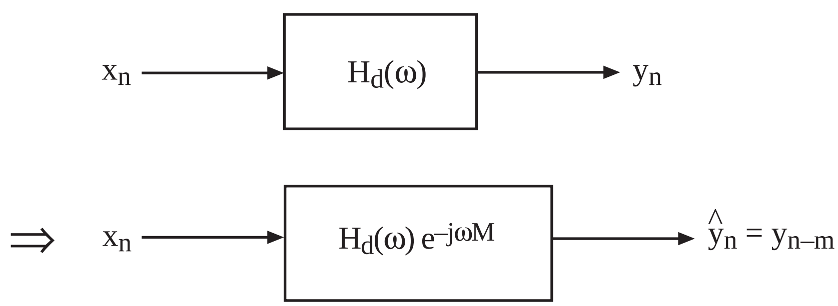
$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega(n-M)}\, d\omega$$

$$= \frac{\omega_c}{\pi} \operatorname{sinc}\left[\omega_c\,(n-M)\right]$$



By truncating this to the left of the origin, we get a causal h_n and this changes $|H_d(\omega)|$ only slightly.

In general, linear phase just adds a delay (which is often acceptable):



Proof:

$$\hat{y}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{H_d(\omega)\, e^{-jM\omega}\, X_d(\omega)}_{\hat{Y}_d(\omega)}\, e^{j\omega n}\, d\omega$$

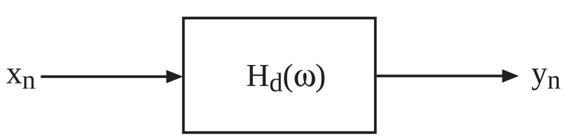
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{H_d(\omega)\, X_d(\omega)}_{Y_d(\omega)}\, e^{j\omega(n-M)}\, d\omega$$

$$= y_{n-M} \quad ✔$$

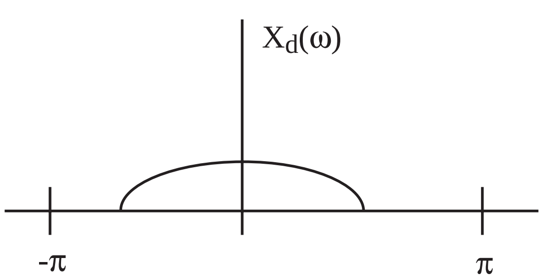
What about nonlinear phase? Answer: Usually don't want it.

Example

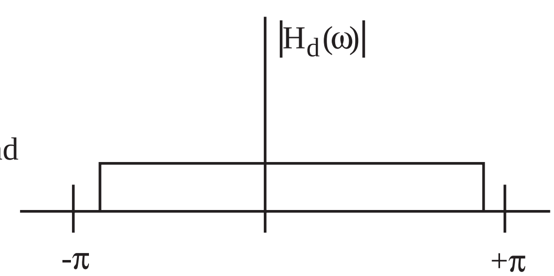
If



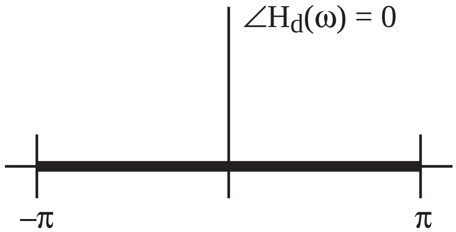
with



and



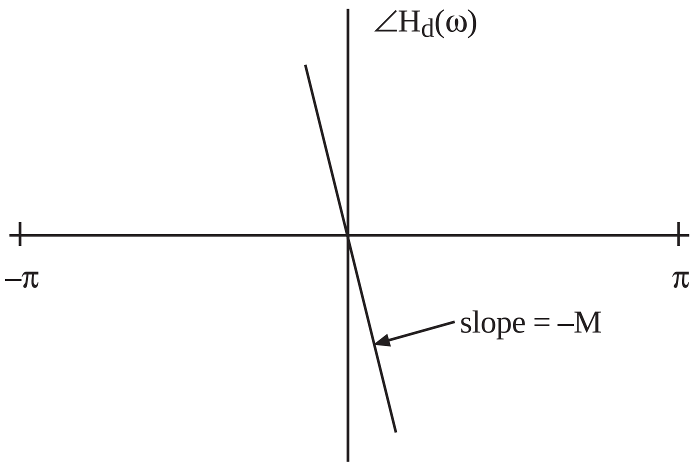
then consider three possibilities for the phase: zero, linear, and nonlinear.



$\Rightarrow$ $H_d(\omega) = |H_d(\omega)|$

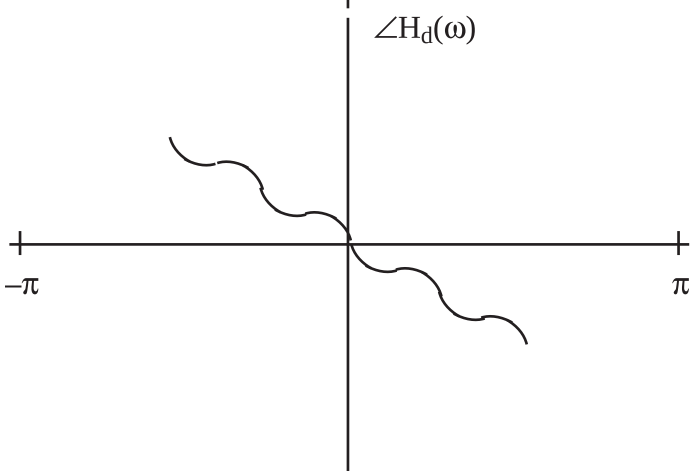
$\Rightarrow$ $Y_d(\omega) = X_d(\omega)$

$\Rightarrow$ $y_n = x_n$



$\Rightarrow$ All frequencies delayed by same amount

$\Rightarrow$ $y_n = x_{n-M}$



$\Rightarrow$ y_n not even close to x_n because different frequencies get delayed by different amounts.

Definition:

Will say $H_d(\omega)$ is <u>linear</u> <u>phase</u> if $H_d(\omega) = \underbrace{|H_d(\omega)|}_{\text{nonnegative}} e^{-j\omega M}$

Comment:

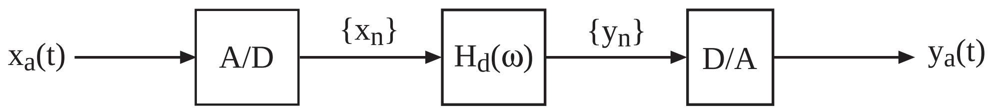
Later in the course, we will consider frequency responses having generalized linear phase where

$$H_d(\omega) = R(\omega)\, e^{-j\omega M}$$

with $R(\omega)$ real-valued, but not necessarily nonnegative.

Analog Frequency Response of a Digital Processor

Consider

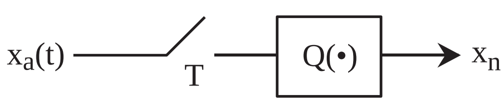


This overall system has an analog input and an analog output. We wish to discover how the analog frequency response depends on $H_d(\omega)$. So, find

$$H_a(\Omega) = \frac{Y_a(\Omega)}{X_a(\Omega)}.$$

We will see that the formula for $Y_a(\Omega)$ in terms of $X_a(\Omega)$ is very complicated, and that in general we can't find this ratio. However, it is possible to find this ratio if we assume that $x_a(t)$ is bandlimited and that we sample above the Nyquist rate. To find $Y_a(\Omega)$ in terms of $X_a(\Omega)$, consider the components in the overall system one at a time, in the frequency domain.

1) A/D

A/D: 

In the analysis, we will neglect the quantizer. Consider

We have shown that:

$$\boxed{X_d(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_a\left(\frac{\omega + 2\pi n}{T}\right)} \qquad (\Diamond)$$

↑

8.16

extremely important

2) Digital Filter

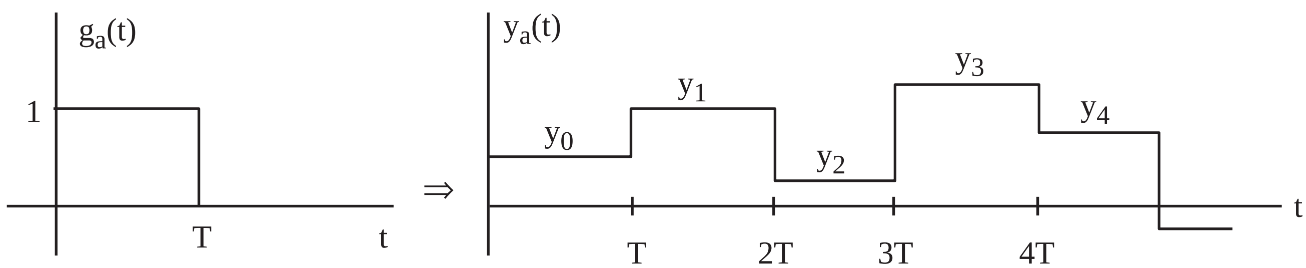
$$Y_d(\omega) = H_d(\omega)\, X_d(\omega)$$

$$\underset{\substack{\uparrow \\ \text{by } (\Diamond)}}{=} \frac{1}{T} H_d(\omega) \sum_{n=-\infty}^{\infty} X_a\left(\frac{\omega + 2\pi n}{T}\right) \qquad (\Diamond\Diamond)$$

3) D/A: Model as

$$y_a(t) = \sum_{n=-\infty}^{\infty} y_n\, g_a(t - nT) \qquad (\square)$$

so that $y_a(t)$ is a weighted pulse train. For example, if $g_a(t)$ is a rectangular pulse then $y_a(t)$ is a staircase function:



We have shown that:

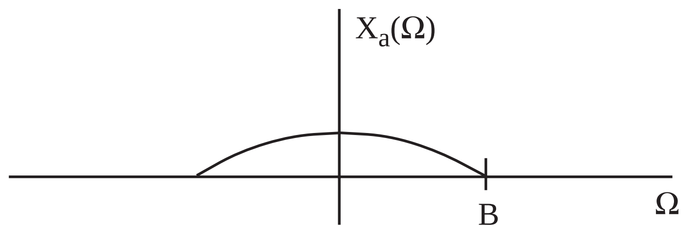
$$\boxed{Y_a(\Omega) = G_a(\Omega)\, Y_d(\Omega T)}$$

Substituting for Y_d from $(\Diamond\Diamond)$ gives us the expression for $Y_a(\Omega)$ in terms of X_a:

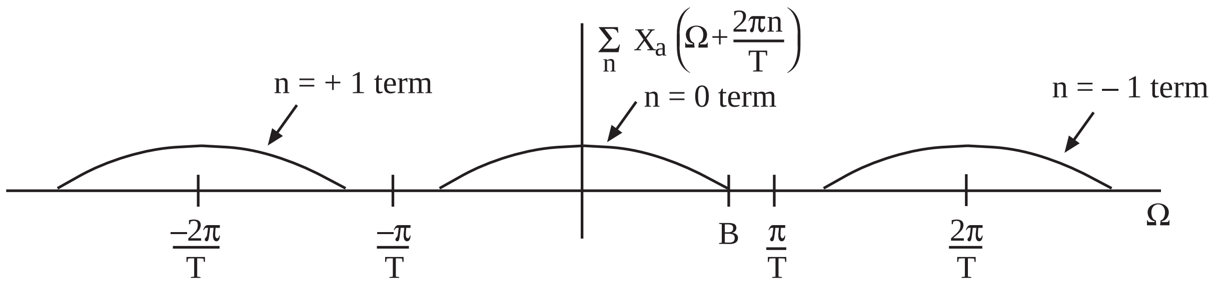
$$\boxed{Y_a(\Omega) = \frac{1}{T} G_a(\Omega) H_d(\Omega T) \sum_{n=-\infty}^{\infty} X_a\left(\Omega + \frac{2\pi n}{T}\right)} \qquad (Ⱶ)$$

Now, take a rest!

This equation is cumbersome and in general we cannot solve for $\frac{Y_a(\Omega)}{X_a(\Omega)}$. Indeed, we cannot define $H_a(\Omega)$ because, although the overall system is linear, in general it is <u>shift-varying</u> so that the system is not describable by a frequency response. Fortunately, (Ⱶ) simplifies tremendously if we assume a bandlimited input with Nyquist-rate sampling, and an ideal D/A converter. Under these conditions the overall system is shift-invariant and it can be described by a frequency response. Let's consider this. Suppose $x_a(t)$ is bandlimited to B rad/sec and we choose $T < \frac{\pi}{B}$. Then, supposing



gives



⇒ No aliasing, so that:

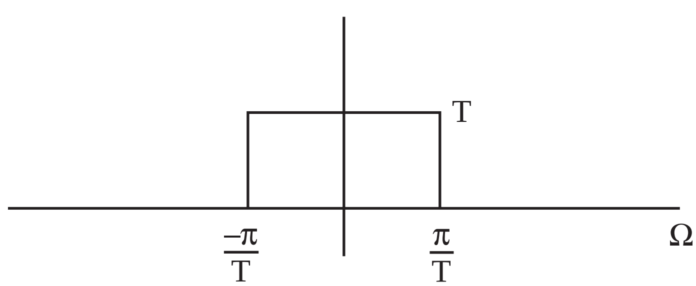
$$\sum_{n=-\infty}^{\infty} X_a\left(\Omega+\frac{2\pi n}{T}\right) = X_a(\Omega), \quad |\Omega| \le \frac{\pi}{T}$$

Now, assume an <u>ideal D/A</u> so that

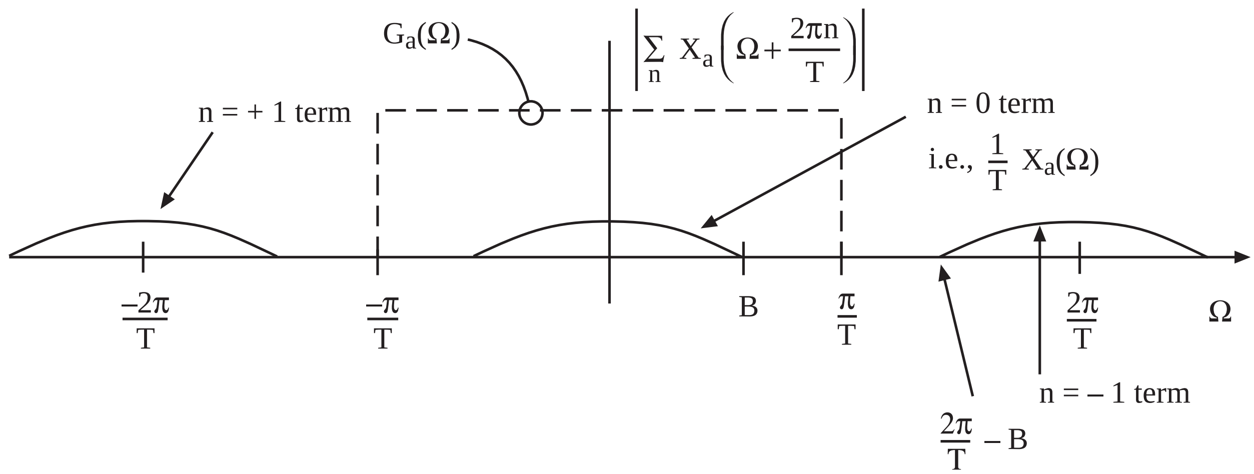
$$g_a(t) = \text{sinc}\,\frac{\pi t}{T} = \frac{\sin\frac{\pi t}{T}}{\frac{\pi t}{T}}$$

Considering the Fourier transform of this pulse, we see that for an ideal D/A, $G_a(\Omega)$ has the shape of an ideal LPF:

$$G_a(\Omega) = \begin{cases} T & |\Omega| \le \frac{\pi}{T} \\ 0 & |\Omega| > \frac{\pi}{T} \end{cases}$$



Now, let's picture the terms that multiply H_d in (ⱶ):



So:

$$G_a(\Omega) \sum_{n=-\infty}^{\infty} X_a\left(\Omega + \frac{2\pi n}{T}\right) = \begin{cases} T \bullet X_a(\Omega) & |\Omega| \le \frac{\pi}{T} \\ 0 & |\Omega| > \frac{\pi}{T} \end{cases}$$

Using this in (ⱶ) gives:

$$Y_a(\Omega) = \begin{cases} H_d(\Omega T) X_a(\Omega) & |\Omega| \le \frac{\pi}{T} \\ 0 & |\Omega| > \frac{\pi}{T} \end{cases}$$

The analog frequency response of the A/D, digital filter, and D/A is

$$H_a(\Omega) = \frac{Y_a(\Omega)}{X_a(\Omega)}$$

$$\Rightarrow \quad \boxed{H_a(\Omega) = \begin{cases} H_d(\Omega T) & |\Omega| \le \frac{\pi}{T} \\ 0 & |\Omega| > \frac{\pi}{T} \end{cases} \qquad (\text{✰})}$$

This is the <u>entire connection</u> between analog and digital filtering! This equation is extremely important!

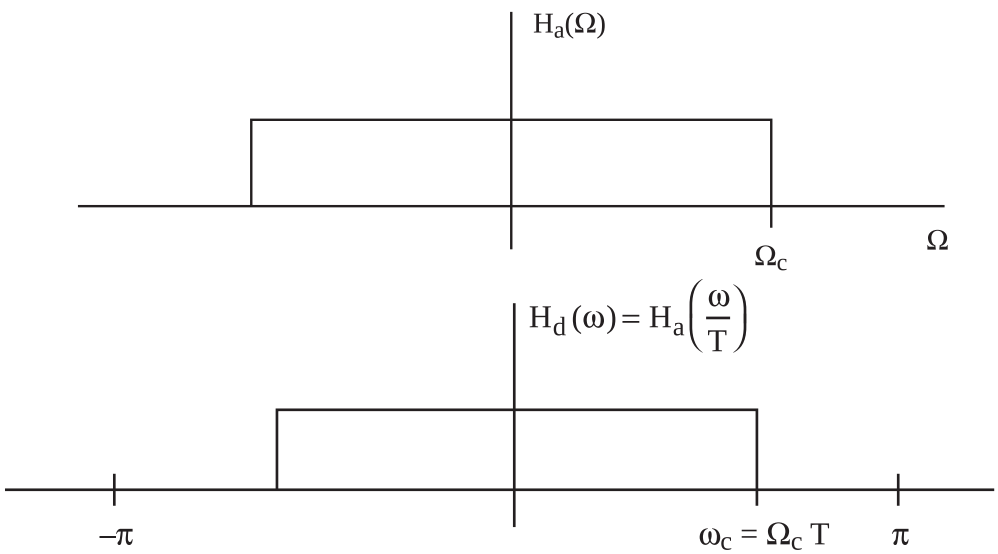
With a change of variable, $\omega = \Omega T$, (✰) becomes:

$$H_a\left(\frac{\omega}{T}\right) = \begin{cases} H_d(\omega) & |\omega| \le \pi \\ 0 & |\omega| > \pi \end{cases}$$

$$\Rightarrow \quad \boxed{H_d(\omega) = H_a\left(\frac{\omega}{T}\right) \qquad |\omega| \le \pi \qquad (☆☆)}$$

So, given a desired $H_a(\Omega)$, $H_d(\omega)$ has the same shape, but on just the center interval $|\omega| \le \pi$.

If the desired analog cutoff frequency is $\Omega = \Omega_c$ how do we choose the digital cutoff?



Remember:

$$\boxed{\omega_c = \Omega_c T}$$

This equation is very handy in specifying cutoff frequencies of digital filters. Likewise, it can be used to find the analog cutoff of a digital system operating with parameters T and ω_c, i.e., $\Omega_c = \omega_c/T$.

Example $x_a(t)$ BL to 50 kHz

Implement analog LPF with

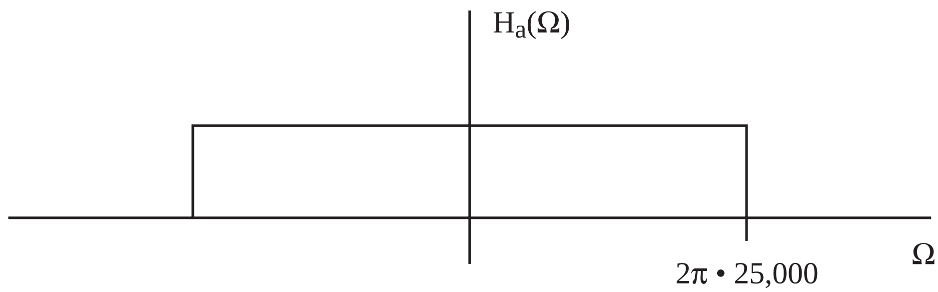
$\Omega_c = 2\pi\ (25{,}000)$ rad/sec.

Choose T according to Nyquist: $\frac{1}{T} = 100{,}000$ samples/sec.

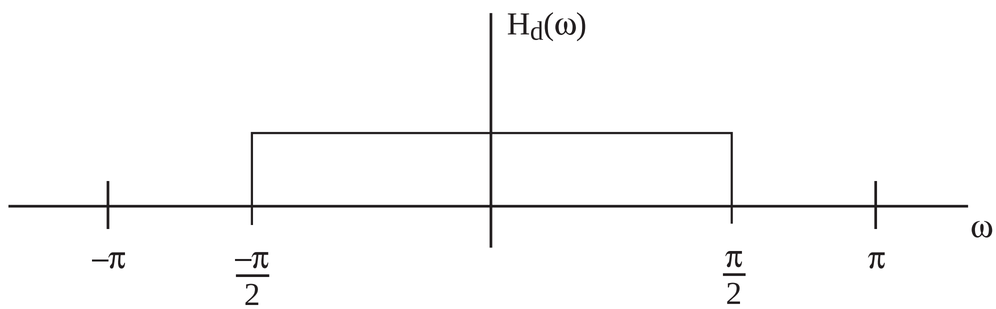
$\Rightarrow T = 10^{-5}$

$\omega_c = \Omega_c T = 2\pi\ (25{,}000)\ 10^{-5} = \frac{\pi}{2}$

So,



is realized by $T = 10^{-5}$ and using



Note: In this example the ratio of the desired analog cutoff to the analog bandwidth was $\frac{1}{2}$. Likewise, the passband of $H_d(\omega)$ filled half of the digital frequency band $|\omega| \le \pi$. This proportional relationship will always hold if we sample at the Nyquist rate.

Question: Why did we sample at the Nyquist rate instead of above it? Why not sample at a rate much greater than 100,000 samples per second? Answer: This would increase hardware cost since it would require a faster A/D and digital filter.

Example

$x_a(t)$ is BL to $2\pi \times 10^6$ rad/sec. Implement analog HPF with $f_c = 250{,}000$ Hz.

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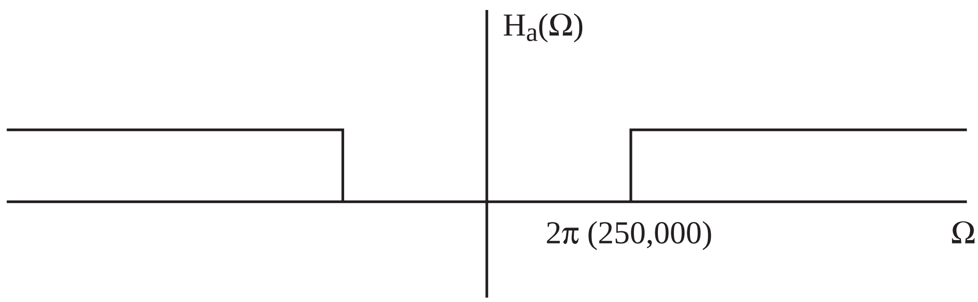
Choose $\frac{1}{T} = 2\left[\frac{2\pi \times 10^6}{2\pi}\right] = 2 \times 10^6$

$$\Rightarrow T = \frac{1}{2 \times 10^6}$$

$$\omega_c = \Omega_c T = 2\pi\,(250{,}000)\,\frac{1}{2 \times 10^6}$$

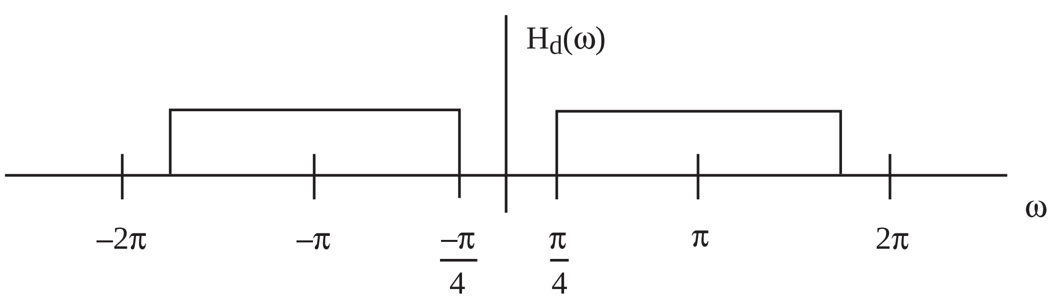
$$= \frac{\pi}{4}$$

So,



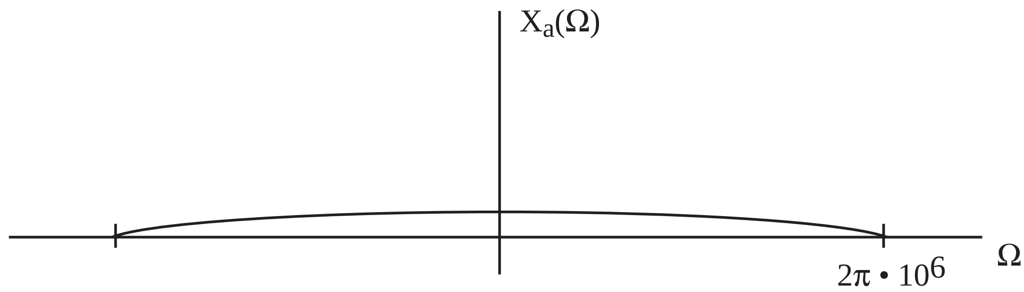
is realized by choosing $T = \frac{1}{2 \times 10^6}$

and using:

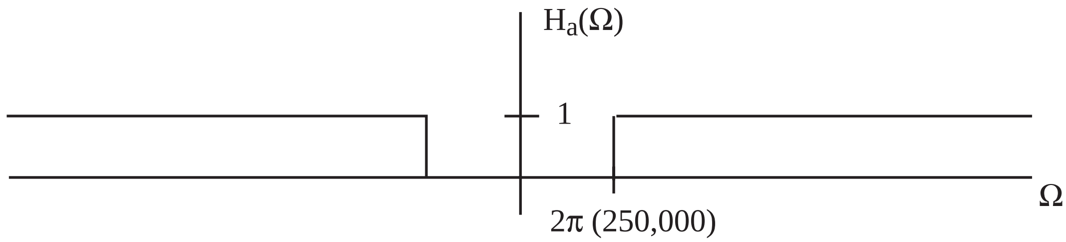


<u>Note</u>: In this situation the <u>overall</u> digital system implements a bandpass filter, but this is equivalent to a HPF because $x_a(t)$ is bandlimited and sampled according to Nyquist.

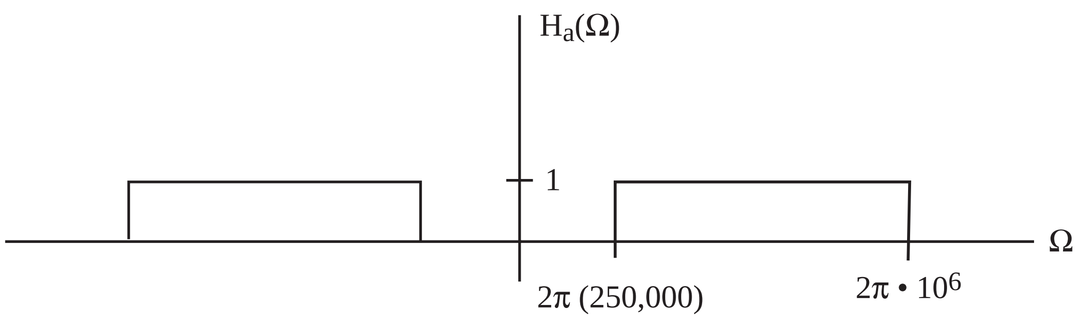
Filtering



with



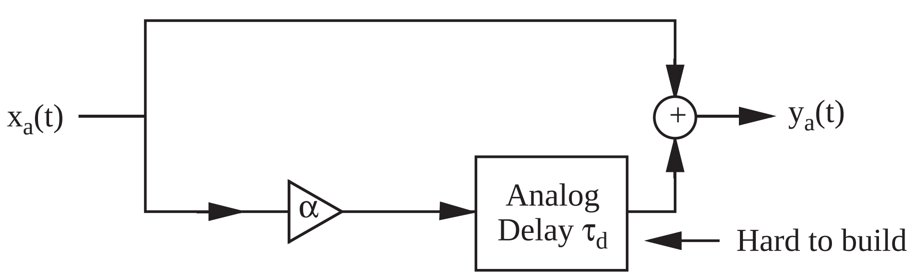
or



gives exactly the same result. The digital system implements the latter (D/A cuts off all frequencies above $2\pi \cdot 10^6$).

Example

Design a digital version of an echo generator:



So, we want a digital system that implements

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$y_a(t) = x_a(t) + \alpha\, x_a(t-\tau_d)$

Assume $x_a(t)$ is bandlimited to 20 kHz.

a) Choose sampling period T.

b) Find desired analog response $H_a(\Omega)$.

c) Find needed digital filter response $H_d(\omega)$.

d) Assuming $\tau_d = kT$, draw a block diagram of the digital filter.

Solution

a) $T < \dfrac{\pi}{\Omega} = \dfrac{\pi}{2\pi \bullet 20\text{ kHz}} = \dfrac{1}{40{,}000}$

Choose

$$\boxed{T = \frac{1}{40{,}000}}$$

b) $Y_a(\Omega) = X_a(\Omega) + \alpha\, X_a(\Omega)\, e^{-j\Omega\tau_d}$

$$\Rightarrow \quad H_a(\Omega) = 1 + \alpha\, e^{-j\Omega\tau_d}$$

c) To find $H_d(\omega)$ in this example, we cannot simply apply $\omega_c = \Omega_c T$, because $H_a(\Omega)$ is not a LPF, HPF, or BFF. There is no Ω_c! Instead, must go back to (☆☆). From (☆☆):

$$H_d(\omega) = H_a\left(\frac{\omega}{T}\right) \qquad |\omega| \le \pi$$

$$= 1 + \alpha\, e^{-j\omega\frac{\tau_d}{T}}$$

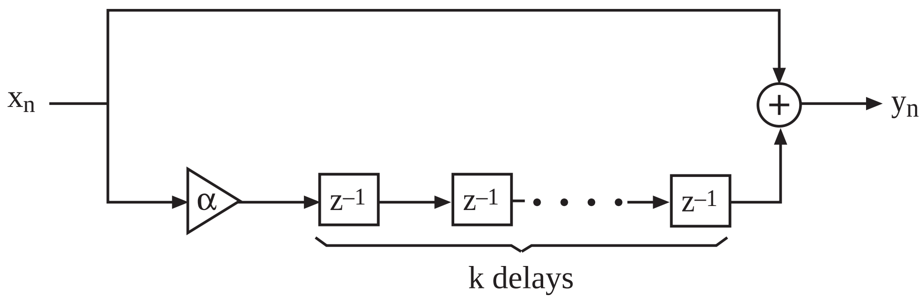
d) If $\tau_d = kT$ then:

$$H_d(\omega) = 1 + \alpha\, e^{-j\omega\frac{kT}{T}}$$

$$= 1 + \alpha\, e^{-jk\omega}$$

$\Rightarrow H(z) = 1 + \alpha\, z^{-k}$

So, the digital filter structure is:



This result completely agrees with our intuition. We could have guessed this!

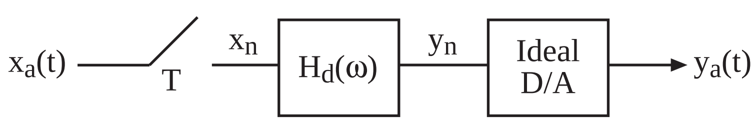
Notes:

1) Using this digital filter between an A/D and D/A implements the desired $H_a(\Omega)$.

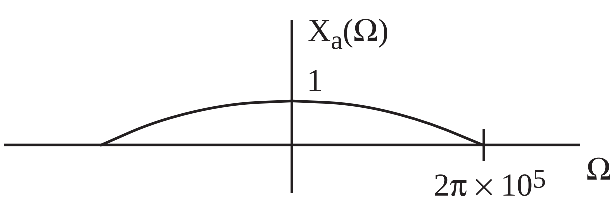
2) If $\tau_d \neq kT$ then H(z) is not a rational function in the variable z^{-1}, and we can only <u>approximate</u> the desired $H_d(\omega)$. Filter design (approximation) will be a major topic later in the course.

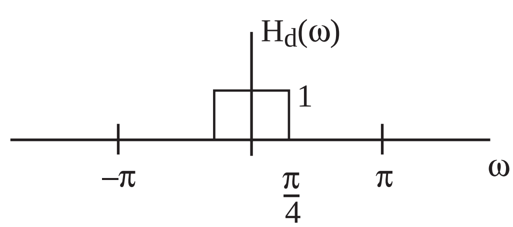
Example

Consider the following system



with $T = \dfrac{1}{3 \times 10^5}$ and





Sketch $X_d(\omega)$, $Y_d(\omega)$, and $Y_a(\Omega)$.

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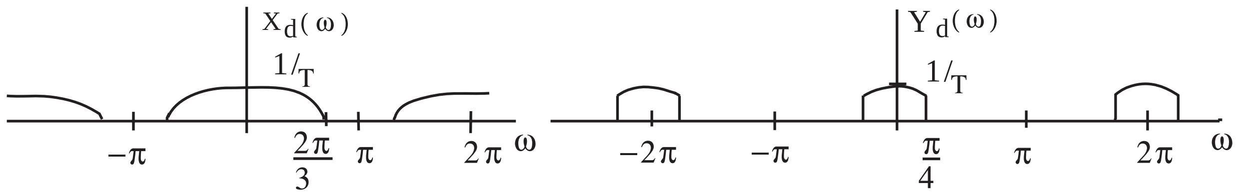
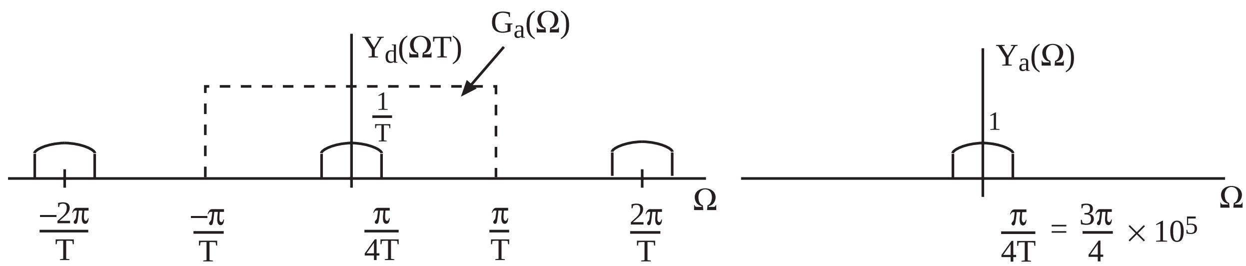
Solution



For an ideal D/A, $Y_a(\Omega) = G_a(\Omega)\, Y_d(\Omega T)$ with $G_a(\Omega) = \begin{cases} T & |\Omega| \le \dfrac{\pi}{T} \\ 0 & \text{else} \end{cases}$. Thus,



Notice that if the D/A is nonideal, and $G_a(\Omega)$ does not cut off abruptly at $\pm \frac{\pi}{T}$, but instead is nonzero for $|\Omega| > \frac{\pi}{T}$, then $Y_a(\Omega)$ will have undesired high-frequency components due to the periodic nature of Y_d. Thus, a critical job of the D/A is to suppress these high-frequency replicas.