# University of Illinois at Urbana-Champaign

# ECE 310: Digital Signal Processing

EXAM 1: SOLUTIONS

*Chandra Radhakrishnan* *Peter Kairouz*

**Problem 1**

(a) True.

(b) False. The statement is true only for LSI systems.

(c) False. Take a cascade of $h_1[n] = u[n]$ and $h_2[n] = 0$, $h[n] = 0$ is bounded. Also, consider $h_1[n] = a^n u[n]$ ($|a| > 1$) and $h_2[n] = \delta[n] - a\delta[n-1]$, $h[n] = \delta[n]$ is bounded.

(d) False. Note this is true if $x[n]$ is real. But for $x[n] = e^{j\pi}$, $y[n]$ will be unbounded.

(e) True.

(f) False. The system is stable since the impulse response is absolutely summable.

(g) False. DTFT is defined for all sequences.

(h) False. Two different finite length sequences will have different DFTs.

(i) False. Note it is a truncated sequence so the frequency response constitutes of two impulses convolved with a sinc function.

(j) False. Consider two inputs $x_1[n]$ and $x_2[n]$. Let $y_1[n]$ and $y_2[n]$ respectively be the outputs for these sequences. Then for an input $x[n] = x_1[n] + x_2[n]$, $y[n] = x[n] + 4 \neq y_1[n] + y_2[n]$

**Problem 2**

(a)

$$
\begin{aligned}
\sum_{k=0}^{7} |X[k]|^2 &= 8 \sum_{n=0}^{7} |x[n]|^2 \\
\sum_{n=0}^{7} |x[n]|^2 &= \sum_{n=0}^{7} \left(\frac{1}{2}\right)^{2n} \\
&= \sum_{n=0}^{7} \left(\frac{1}{4}\right)^{n} \\
&= \frac{1 - \left(\frac{1}{4}\right)^8}{1 - \frac{1}{4}} \\
\Rightarrow \sum_{k=0}^{7} |X[k]|^2 &= 8 \frac{1 - \left(\frac{1}{4}\right)^8}{1 - \frac{1}{4}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} \\
&= \sum_{n=0}^{7} x[n] e^{-j\frac{2\pi kn}{8}} \\
&= \sum_{n=0}^{7} x[n] e^{-j\frac{\pi kn}{4}} \\
&= \sum_{n=0}^{7} \left(\frac{1}{2}\right)^n e^{-j\frac{\pi kn}{4}} \\
&= \sum_{n=0}^{7} \left(\frac{1}{2} e^{-j\frac{\pi k}{4}}\right)^n \\
&= \frac{1-\left(\frac{1}{2}\right)^8 e^{-j2\pi k}}{1-\frac{1}{2}e^{-j\frac{\pi k}{4}}} = \frac{1-\left(\frac{1}{2}\right)^8}{1-\frac{1}{2}e^{-j\frac{\pi k}{4}}}
\end{aligned}
$$

**Problem 3**

(a)

$$
\begin{aligned}
X_d(0) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}|_{\omega=0} \\
&= \sum_{n=-\infty}^{\infty} x[n] \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^{2k} + \left(-\frac{1}{4}+\frac{1}{4}+\frac{3}{4}+\frac{5}{4}\right) \\
&= \frac{1}{1-\frac{1}{2}} + 2 \\
&= 4
\end{aligned}
$$

(b)

$$
\begin{aligned}
X_d(\pi) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}|_{\omega=\pi} \\
&= \sum_{n=-\infty}^{\infty} x[n](-1)^n \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^{2k} - \left(-\frac{1}{4}+\frac{1}{4}+\frac{3}{4}+\frac{5}{4}\right) \\
&= 2-2=0
\end{aligned}
$$

(c)

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} |x[n]|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X_d(\omega)|^2 d\omega \\
\Rightarrow \int_{-\pi}^{\pi} |X_d(\omega)|^2 d\omega &= 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 \\
&= 2\pi \sum_{n=-\infty}^{\infty} \left|\left(\frac{1}{\sqrt{2}}\right)^{2n}\right|^2 + \left(\frac{1}{16} + \frac{1}{16} + \frac{9}{16} + \frac{25}{16}\right) \\
&= 2\pi \left[\left(\frac{1}{4}\right)^n + \frac{1}{16}\frac{9}{4}\right] \\
&= 2\pi \left(\frac{3}{4} + \frac{9}{4}\right) \\
&= 2\pi * \frac{43}{12} \\
&= \frac{43}{6}\pi
\end{aligned}
$$

(d)

$$
\int_{-\pi}^{\pi} |Y_d(\omega)|^2 d\omega = 2\pi y[0]
$$

$y[0]$ can be evaluated as follows,

$$
\begin{aligned}
y[n] &= h[n] * x[n] \\
&= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\
&= \sum_{n=-\infty}^{\infty} x[k]x[k-n] \\
\Rightarrow 2\pi y[0] &= 2\pi \sum_{n=-\infty}^{\infty} |x[k]|^2 \\
&= \int_{-\pi}^{\pi} |X_d(\omega)|^2 d\omega = \frac{43}{6}\pi
\end{aligned}
$$

**Problem 4**

a. Characteristic Equation :

$$
\begin{aligned}
z^2 - 8z + 16 &= 0 \\
(z-4)^2 &= 0 \\
z &= 4, 4
\end{aligned}
$$

$y[n]$ can now be written as,

$$
y[n] = A(4)^n + Bn(4)^n
$$

b. Using initial conditions we have,

$$\begin{aligned} A - B &= 2 \\ A - 2B &= 1 \end{aligned}$$

Solving for $A$ and $B$ we have $A = 3, B = 1$. The solution can be written as,

$$y[n] = (n+3)4^n, n \geq -2$$

**Problem 5**

a. We must have

$$z^2 + 1 = 0$$

Which yields,

$$z = \pm j$$

$y[n]$ can now be written as

$$y[n] = A(j)^n + B(-j)^n$$

b. Using initial conditions,

$$\begin{aligned} y(-1) &= 4 = -jA + jB \\ y(-2) &= 0 = -A - B \end{aligned}$$

Hence we have

$$\begin{aligned} A &= -B \\ 4 &= 2jB \end{aligned}$$

which yield

$$\begin{aligned} A &= j2 \\ B &= -j2 \end{aligned}$$

c. Difference equation is valid for $n \geq 0$ For $n = 0, y(-5) = -y(-3)$
For $n = 1, y(-4) = -y(-2)$
For $n = 2, y(-3) = -y(-1)$
Hence given $y(-1), y(-2)$, the solution holds for $n \geq -5$.

d. $y(21)$ can be computed as shown below,

$$\begin{aligned} y(n) &= 2(j)^{n+1} + 2(-j)^{n+1}, n \geq -5 \\ y(21) &= 2\left(j^{22} + (-j)^{22}\right) \\ &= 2[-1-1] = -4 \end{aligned}$$

**Problem 6**

a. Note that, $\omega = \frac{2\pi k}{21}$

(a) $X_d^*\left(\frac{14\pi}{21}\right) = X[-7]$, Hence False.

(b) True

(c) $X_d^*\left(\frac{4\pi}{7}\right) = X_d^*\left(\frac{12\pi}{21}\right) = X_d^*\left(\frac{2\pi 6}{21}\right)$. Hence False.

(d) $X_d^*\left(\frac{-4\pi}{21}\right) = X^*(-2)$. Hence True.

(e) $X_8 \neq 0$

(f) $X_d^*\left(\frac{18\pi}{21}\right) = X^*(9) = X(-9)$

b. Let $f_1[n] = \{2, 0, 6, 4\}$ and $f_2[n] = \{2, 1, 0, 3\}$ Then,

$$\begin{aligned} DFT\{f_2[n]\} &= \frac{1}{2}DFT\{f_1[< n-1 >_4]\} \\ &= \frac{1}{2}DFT\{f_1[n]\}e^{-j\frac{2\pi}{4}} \end{aligned}$$

Hence we have,

$$\begin{aligned} F_2[k] &= \left\{\frac{1}{2}X_0e^{-j0}, \frac{1}{2}X_1e^{-j\frac{\pi}{2}}, \frac{1}{2}X_2e^{-j\pi}, \frac{1}{2}X_3e^{-j\frac{3\pi}{2}}\right\} \\ &= \left\{\frac{X_0}{2}, \frac{-jX_1}{2}, \frac{-X_2}{2}, \frac{jX_3}{2}\right\} \end{aligned}$$

c. The DTFT of $y[n]$, $Y[k]$ can computed as follows,

$$\begin{aligned} Y[k] = DTFT\left(\mathcal{R}\left(x^*[n]\right)\right) &= DTFT\left\{\frac{1}{2}\left(x[n] + x^*[n]\right)\right\} \\ &= \frac{1}{2}DTFT\left\{x[n]\right\} + \frac{1}{2}DTFT\left\{x^*[n]\right\} \\ &= \frac{X(\omega) + X^*(-\omega)}{2} \end{aligned}$$

d. Note this problem had an error. DTFT of $x[n]$ is $X_d(\omega)$. The DTFT of $x[n/L]$ of the sequence $y[n]$ where,

$$y[n] = \begin{cases} x[n/L], & \text{if } n \text{ is an integer multiple of } L \\ 0, & \text{otherwise,} \end{cases}$$

is given by,

$$\begin{aligned} Y(\omega) &= \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} \\ &= \sum_{n=mult.of L} y[n]e^{-j\omega n} \\ &= \sum_{k=-\infty}^{\infty} y[kL]e^{-j\omega kL} \\ &= X(\omega L) \end{aligned}$$

**Problem 7**

a. The system $\mathcal{S}$ is known to be LTI. Hence we have,

$$\delta[n] = x_1[n] - 2x_2[n]$$

which gives the impulse response as,

$$\begin{aligned} h[n] &= y_1[n] - 2y_2[n] \\ &= [0, 1, -1] \end{aligned}$$

b. Note that the impulse response and the input sequence are of length 3. Hence the convolution result will be of length 5. Flipping $h[n]$ and performing convolution the usual way will yield,

$$y[n] = x[n] * h[n] = [1, -2, 2, -1, 0]$$

**Problem 8**

The impulse response is given by $h[n] = (-1)^n u[n]$

a. The system is BIBO stable if

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

clearly,

$$\sum_{n=-\infty}^{\infty} |(-1^n)u[n]|$$

is not absolutely summable. For $x[n] = (-1)^n u[n]$, the output will be unbounded.

b. The output of the system is given by,

$$\begin{aligned} y[n] &= (x[n] + x[n] * h_1[n]) * h_2[n] \\ &= \{x[n] * (\delta[n] + h_1[n])\} * h_2[n] \end{aligned} \tag{1}$$

Let $h[n]$ denote the impulse response of the overall system then,

$$y[n] = x[n] * h[n]. \tag{2}$$

Comparing (1) and (2) we have,

$$\begin{aligned} h[n] &= (\delta[n] + h_1[n]) * h_2[n] \\ &= h_2[n] + h_1[n] * h_2[n] \\ &= \alpha^n u[n] + \beta\alpha^{n-1}u[n-1] \end{aligned}$$

c. This is a cascade of two systems given by $H_1(z) = 1 + \beta z^{-1}$ and $H_2(z) = \frac{z}{z-\alpha}$. The overall cascade of the systems is given by,

$$\begin{aligned} H(z) = \frac{Y(z)}{X(z)} &= (1 + \beta z^{-1}) \left(\frac{z}{z-\alpha}\right) \\ &= \frac{z+\beta}{z-\alpha} \end{aligned}$$

Hence we have,

$$\begin{aligned} Y(z)(z-\alpha) &= X(z)(z+\beta) \\ \text{or, } Y(z)(1-\alpha z^{-1}) &= X(z)(1+\beta z^{-1}) \\ \Rightarrow y[n] &= \alpha y[n-1] + x[n] + \beta x[n-1] \end{aligned}$$

(d) It can be easily seen that system is causal. The system is stable if $|\alpha| < 1$ and $\beta < \infty$.

**Problem 9**

1. The ROC does not include the unit circle so the DTFT does not exist, hence (d)
2. Substitute $z = e^{j\omega}$ which yields (a).
3. For a real valued sequence $X_d(\omega) = X_d^*(-\omega)$, hence (a).
4. Using the fact that DTFT is periodic with period $2\pi$ we get (a).

**Problem 10** We have,

$$\begin{aligned} h[n] * h_{inv}[n] &= \delta[n] \\ \Rightarrow H(z)H_{inv}(z) &= 1 \\ \Rightarrow H_{inv}(z) &= \frac{1}{H(z)} \end{aligned}$$

The $z$-transform of $h[n] = (3^{-n} + 2^{-n})u[n]$ is,

$$\begin{aligned} H(z) &= \frac{z}{z - \frac{1}{3}} + \frac{z}{z - \frac{1}{2}} \\ &= \frac{z}{z - \frac{1}{3}} + \frac{z}{z - \frac{1}{2}}, \quad \text{ROC:}|z| > \frac{1}{2} \\ &= \frac{z(z - \frac{1}{2}) + z(z - \frac{1}{3})}{(z - \frac{1}{3})(z - \frac{1}{2})} \\ &= \frac{2z^2 - \frac{5}{6}z}{(z - \frac{1}{3})(z - \frac{1}{2})} \\ &= \frac{2z\left(z - \frac{5}{12}\right)}{(z - \frac{1}{3})(z - \frac{1}{2})} \end{aligned}$$

Therefore $H_{inv}(z)$ can be computed as,

$$\begin{aligned} H_{inv}(z) &= \frac{1}{H(z)} \\ &= \frac{(z - \frac{1}{3})(z - \frac{1}{2})}{2z\left(z - \frac{5}{12}\right)} \end{aligned}$$

In the exam full credit was given for computing the inverse system shown above. The system $H_{inv}(z)$ has poles at $z = 0, \frac{5}{12}$. We can have two choices for ROC : $|z| > \frac{5}{12}$ or $0 < |z| < \frac{5}{12}$. The ROC of an inverse system must overlap with the ROC of the original system $H(z)$. Therefore we have,

$$H_{inv}(z) = \frac{(z - \frac{1}{3})(z - \frac{1}{2})}{2z\left(z - \frac{5}{12}\right)}, \quad \text{ROC:}|z| > \frac{5}{12}$$

**Problem 11**

(a) For $T = \frac{1}{3 \times 10^3}$ we have,

$$\Omega_s = \frac{2\pi}{T} = 6\pi \times 10^3$$

The sketch of $X_d(\omega)$ for $T = \frac{1}{3 \times 10^3}$ is shown in Fig. 1. For $T = \frac{1}{8 \times 10^3}$ we have,

$$\Omega_s = \frac{2\pi}{T} = 16\pi \times 10^3$$

The sketch of $X_d(\omega)$ for $T = \frac{1}{8 \times 10^3}$ is shown in Fig. 2.

(b) The result of applying $H_d(\omega)$ we is shown in Fig. 3

(c) Note that $X_d(\Omega)$ must be shifted right by $\frac{3\pi}{8}$. Hence $\omega_0 = \frac{3\pi}{8}$.

**Problem 12**

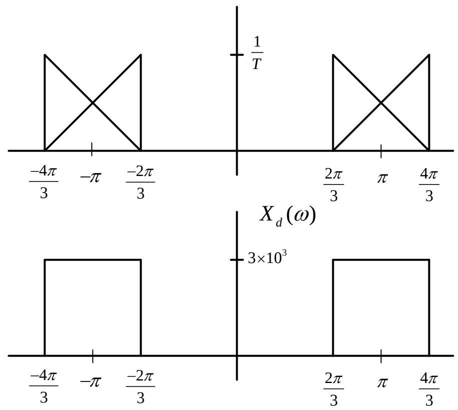


Figure 1: $X_d(\omega)$ for $T = \frac{1}{3\times10^3}$

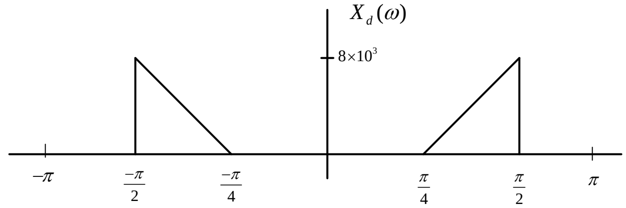


Figure 2: $X_d(\omega)$ for $T = \frac{1}{8\times10^3}$

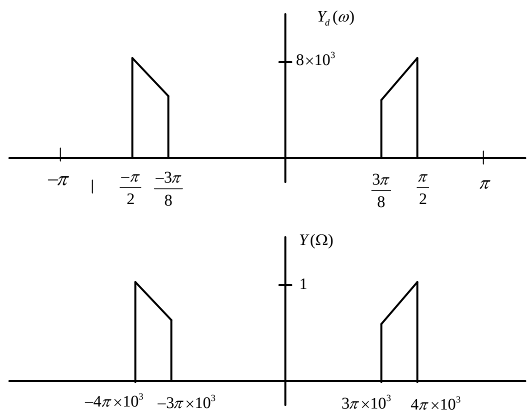


Figure 3: $Y_d(\omega)$ and $Y_a(\Omega)$ for Problem 11b.

(a) Consider the following three cases
Case I:$-3 \leq n \leq 1$

$$
\begin{aligned}
y[n] &= \sum_{k=-3}^{n} \alpha^k \\
&= \frac{\alpha^{-3} - \alpha^{n+1}}{1-\alpha} \left(u[n+3] - u[n-2]\right)
\end{aligned}
$$

Case II: $2 \leq n \leq 5$

$$\begin{aligned} y[n] &= \sum_{n-4}^{2} \alpha^k \\ &= \frac{\alpha^{n-4} - \alpha^{n+1}}{1 - \alpha} \\ &= \alpha^n \frac{\left(\alpha^{-4} - \alpha\right)}{1 - \alpha} \left(u[n-2] - u[n-6]\right) \end{aligned}$$

Case III: $6 \leq n \leq 9$

$$\begin{aligned} y[n] &= \sum_{n-4}^{5} \alpha^k \\ &= \frac{\alpha^{n-4} - \alpha^6}{1 - \alpha} \left(u[n-6] - u[n-10]\right) \end{aligned}$$

(b)

$$\begin{aligned} \mathcal{Z}(x[n] * h[n]) = Y(z) &= \mathcal{Z}(x[n])\mathcal{Z}(h[n]) \\ &= \sum_{n=-3}^{5} \alpha^n z^{-n} \sum_{k=0}^{4} z^{-k} \\ &= \sum_{n=-3}^{5} \sum_{k=0}^{4} \alpha^n z^{-(n+k)} \end{aligned}$$

(c) Note,

$$\begin{aligned} \mathcal{Z}(x[n-2]) &= z^{-2} X(z) \\ \mathcal{Z}(x[n-3]) &= z^{-3} Y(z) \\ \Rightarrow x[n-2] * h[n-3] &= y[n-5] \end{aligned}$$

(d) $\mathcal{Z}(x[n-2] * h[n-3]) = z^{-5} Y(z)$