Problem 1 (10 points)

Mark True or False for the following statements. You must justify your answer.

a. A BIBO stable LSI system with transfer function $H(z) = \frac{(z-1)(z+1)(z-6)}{(z-2)^2(z+\frac{1}{2})}$ could correspond to a **causal** system. **T/F**

b. The zero-state response $y[n]$ of a BIBO unstable LSI system to any non-zero input $x[n]$ can be bounded. **T/F**

c. An input $x[n] = cos\omega_0 n$ to the LSI system $H_d(\omega)$ produces an output of the form $|H_d(\omega)|x[n]e^{\angle H_d(\omega_0)}$. **T/F**

d. The windowing operation causes the phase of an FIR filter to be non-zero. **T/F**

e. It is not possible to design a low-pass FIR filter whose impulse response has anti symmetry. **T/F**

(a) False: poles at $z=2$, $z=-\frac{1}{2}$

ROC: for stable system must include unit circle

$\Rightarrow$ System cannot be causal.

(b) There was ambiguity in the problem. "Any input" could mean "all possible i/p", in which case the answer is FALSE. "Any input" could mean "one particular in input" in which case answer is TRUE. (In exam credit was given as long as the answer was consistent with the justification

(c) FALSE: Note: $\cos\omega_0 n$ is not an eigenfunction

(d) FALSE: The shifting causes linear phase

(e) TRUE: Since $H_d(0) = 0$.

Problem 2 (30 points)

Consider the z-transform $X(z)$ whose pole-zero plot is as shown in Fig. 1.

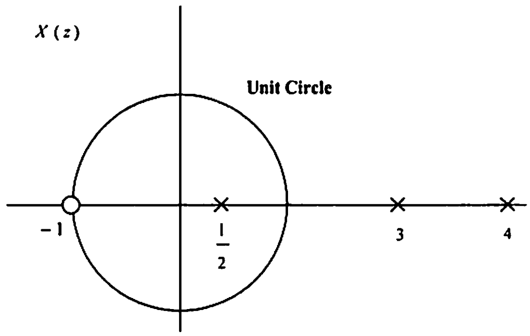


Figure 1: Pole-zero plot for Problem 2

(a) Determine the region of convergence of $X(z)$ if it is known that the Fourier transform exists. For this case, determine whether the corresponding sequence $x[n]$ is right sided, left sided, or two sided.

(b) Given that an one sided sequence has the specified pole zero plot. Will the Fourier transform of $x[n]$ exist?

(c) How many possible two sided sequences can have the pole zero plot shown in Fig. 1? Specify ROC.

(d) How many possible one sided sequences can have the pole zero plot shown in Fig. 1? Specify ROC.

(e) Is it possible for the pole-zero plot in Fig. 1 to be associated with a sequence that is both causal and stable? If so, specify the appropriate region of convergence.

(a) Fourier transform exist $\Rightarrow$ Roc includes unit circle

$\frac{1}{2} < |z| < 3$

(b) Possible Roc of one sided sequence

(1) $|z| > 4$ or (2) $|z| < \frac{1}{2}$

$\Rightarrow$ Roc does not contain unit circle $\Rightarrow$ FT does not exist

(c) There are two possible 2 sided sequences

Roc's : $\frac{1}{2} < |z| < 3$ or Roc : $3 < |z| < 4$

(d) There are two possible 1 sided sequences

Roc : $|z| > 4$ or $|z| < \frac{1}{2}$

(e) For stability Roc must contain unit circle.

For causality the Roc must be outside the outermost pole

$\therefore$ These conditions cannot be met by any possible Roc.

Problem 3 (10 points)

The pole-zero diagram in Fig. 1 corresponds to the z-transform $X(z)$ of a causal sequence $x[n]$. Sketch the pole=zero diagram of $Y(z)$, where $y[n] = x[-n+3]$. Also, specify the ROC for $Y(z)$.

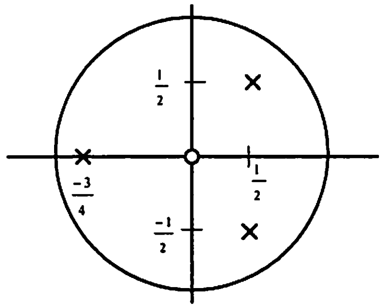


Figure 1: Pole-zero plot for Problem 3

From pole - zero diagram :

$$X(z) = \frac{z}{\left(z^2 - z + \frac{1}{2}\right)\left(z + \frac{3}{4}\right)} \qquad |z| > \frac{3}{4}$$

given : $y[n] = x[-n+3]$

$$Y(z) = \sum_{n=-\infty}^{\infty} x[-n+3]\, z^{-n}$$

put $t = -n+3$

$$\Rightarrow Y(z) = \sum_{t=-\infty}^{\infty} x[t]\, z^{t-3}$$

$$= z^{-3} \sum_{t=-\infty}^{\infty} x(t)\, z^{t}$$

$$= z^{-3} \sum_{t=-\infty}^{\infty} x(t)\left(z^{-1}\right)^{-t}$$

$$\Rightarrow Y(z) = z^{-3} X(z^{-1})$$

Therefore : $Y(z) = \dfrac{z^{-3} \quad z^{-1}}{\left(z^{-2} - z^{-1} + \frac{1}{2}\right)\left(z^{-1} + \frac{3}{4}\right)}$

$$= \frac{8/3}{z\,(2 - 2z + z^2)\left(\frac{4}{3} + z\right)}$$

Poles at : $z = 0 \qquad z = -\frac{4}{3} \qquad z = 1 \mp j$

$x[n]$ is causal $\Rightarrow$ $x[-n+3]$ is left sided

$\Rightarrow$ ROC : $0 < |z| < \frac{4}{3}$

Problem 4 (30 points)

Consider a 5-tap generalized linear-phase FIR filter which satisfies the following conditions:

- The operation $y[n] = h[n] * u[n]$ yields $y[0] = 4$, where $*$ denotes convolution.
- The DC component of the input is amplified six times.
- A sinusoidal input at the highest digital frequency is amplified six times.

where $*$ represents convolution operation, $u[n]$ the unit step input, and $H_d(\omega)$ is the frequency response.

(a) Is the filter Type I or Type II ?

(b) Find the impulse response, $h[n]$?

(c) Sketch the magnitude and phase responses of $H_d(\omega)$. Note: The frequency points on the x-axis must be clearly specified to get credit for this part.

(d) Draw the tapped-delay line structure of this FIR Filter.

(a) The filter is Type I since $H_d(0) \neq 0$

(b) $y[0] = 4$

$\Rightarrow \boxed{h_0 = 4}$

$$H_d(\pi) = \sum_{n=0}^{4} h[n](-1)^n = 2h_0 - 2h_1 + h_2 = 6 \quad -①$$

$$H_d(0) = \sum_{n=0}^{4} h[n] = 2h_0 + 2h_1 + h_2 = 6 \quad -②$$

Solving ①, ②,

$h_1 = 0, \quad h_2 = -2$

Hence the filter coefficients are $[4 \; 0 \; -2 \; 0 \; 4]$

(c) $H_d(\omega) = 4 - 2e^{-j2\omega} + 4e^{-j4\omega}$

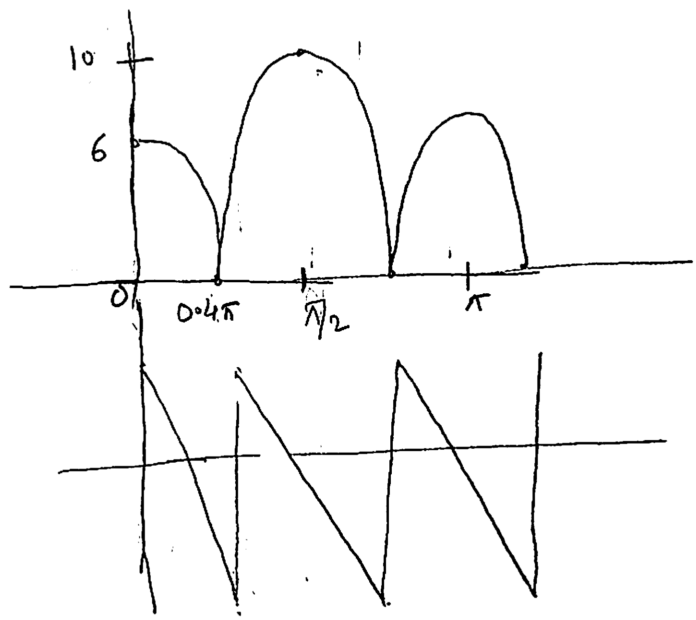
$$H_d(\omega) = e^{-j2\omega}\left[4e^{j2\omega} - 2 + 4e^{-j2\omega}\right]$$

$$= e^{-j2\omega}\left[-2 + 4e^{j2\omega} + 4e^{-j2\omega}\right]$$

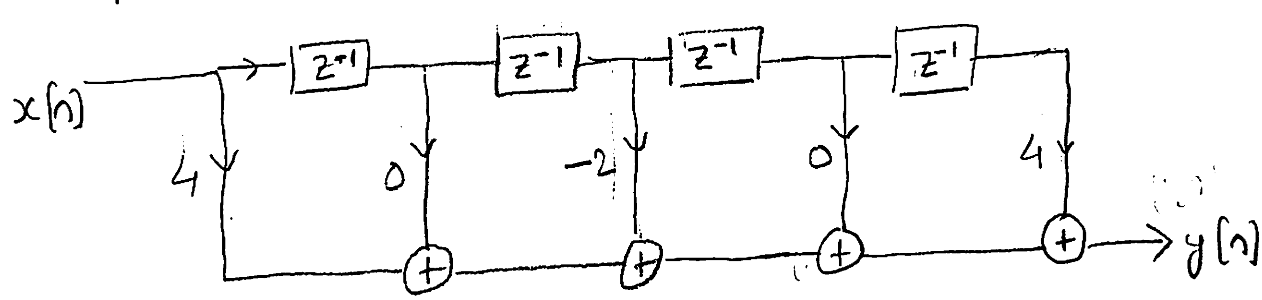
$$= e^{-j2\omega}\left[-2 + 8\cos 2\omega\right]$$

$$\Rightarrow \quad \angle H_d(\omega) = -2\omega$$

$$|H_d(\omega)| = |-2 + 8\cos 2\omega|$$



(d) Tapped delay structure :



Problem 5 (20 points)

The frequency response of a system is given below,

$$|H_d(\omega)| = \begin{cases} 0.7, & -\pi \le \omega < 0 \\ \frac{1}{4}\omega, & 0 \le \omega < \frac{\pi}{4} \\ \frac{1}{2}, & \frac{\pi}{4} \le \omega < \frac{\pi}{2} \\ 1, & \frac{\pi}{2} \le \omega \le \pi \end{cases}$$

$$\angle H_d(\omega) = \begin{cases} 2(-\frac{\pi}{2} - \omega), & -\pi \le \omega \le \frac{-\pi}{2} \\ 0, & -\frac{\pi}{2} < \omega < \frac{\pi}{2} \\ 2(-\frac{\pi}{2} + \omega), & \frac{\pi}{2} \le \omega \le \pi \end{cases}$$

(a) Determine the output of the system for the following input,

$$x[n] = cos(\frac{\pi n}{6}) + e^{j\frac{\pi n}{3}} + cos(\frac{2\pi n}{3}) + cos(\frac{3\pi n}{2})$$

(b) Consider the sequence $x[n]$ such that

$$x[n] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & otherwise \end{cases}$$

Let $\tilde{x}[n]$ be periodic version of $x[n]$,

$$\tilde{x}[n] = \sum_{k=-\infty}^{\infty} x[n + 10k]$$

Given that the frequency response $H_d(\omega)$ of an LSI system is,

$$H_d(\omega) = 1, \quad |\omega| \le \frac{2\pi}{10}$$

Determine the output of the system when $\tilde{x}[n]$ is the input.

(a)

① $\cos \frac{\pi n}{6} \Rightarrow \omega = \frac{\pi}{6} \;\therefore\; 0 \le \omega < \pi/4$

$\Rightarrow \cos \frac{\pi n}{6} \to \frac{1}{4} \cdot \frac{\pi}{6} \cos\left(\frac{\pi n}{6}\right). = \frac{\pi}{24} \cos \frac{\pi n}{6}$

② $e^{j\pi n/3} \Rightarrow \omega = \pi/3 \Rightarrow \frac{\pi}{4} < \omega < \pi/2$

$e^{j\pi n/3} \to \frac{1}{2} e^{j\frac{\pi n}{3}}$

③ $\cos \frac{2\pi n}{3} \Rightarrow \omega = \frac{2\pi}{3}$; $\frac{\pi}{2} < \omega < \pi$

$$\angle H_d(\omega) = 2\left(-\frac{\pi}{2} + \frac{2\pi}{3}\right) = -\frac{\pi}{3}$$

$$\Rightarrow \cos\frac{2\pi n}{3} \rightarrow \cos\left(\frac{2\pi n}{3} - \frac{\pi}{3}\right)$$

④ $\cos\frac{3\pi n}{2} = \cos\left(\pi n + \frac{\pi}{2}n\right) = -\cos\frac{\pi}{2}n$

$$\angle H_d(\omega) = 2\left(-\frac{\pi}{2} + \frac{\pi}{2}\right) = 0$$

$$\Rightarrow \cos 3\frac{\pi n}{2} \rightarrow 0.7\cos\frac{\pi}{2}n$$

Hence $y[n] = \frac{\pi}{24}\cos\frac{\pi n}{6} + \frac{1}{2}e^{j\pi n/3} + \cos\left(\frac{2\pi n}{3} - \frac{\pi}{3}\right) - 0.7\cos\frac{\pi}{2}n$

⑤ ⓑ The DFS of $\tilde{x}(n)$ is,

$$\tilde{X}[k] = e^{-j\left(\frac{4\pi k}{10}\right)} \cdot \frac{\sin(\pi k/2)}{\sin \pi k/10} \qquad 0 \leq k \leq N-1$$

here $N = 10$

$$\Rightarrow \tilde{x}[n] = \frac{1}{10}\sum_{k=0}^{9}\tilde{X}[k]\, e^{j\frac{2\pi k n}{10}}$$

Since $H_d(\omega) = 1 \quad |\omega| < \frac{2\pi}{10}$

$$y[n] = \frac{1}{10}X(0)$$

Problem 6 (20 points)

A complex-valued continuous-time signal $x_a(t)$ has the Fourier transform shown in Fig. 1, where $\Omega_1 = 2 \times \pi \times 4000$ rad/s and $\Omega_2 = 2 \times \pi \times 6000$ rad/s.

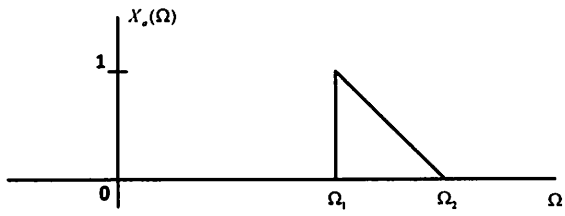


Figure 1: Fourier transform for signal in Problem 6

(a) What is the Nyquist sampling rate for this signal? Sketch the spectrum of $X_d(\omega)$ when the signal is sampled at the Nyquist rate.

(b) What is the **lowest** sampling frequency that can be used without incurring any aliasing distortion, i.e., so that $x_a(t)$ can be recovered from $x[n]$. Sketch the spectrum of the signal obtained by sampling at the lowest sampling rate.

(c) Consider the block diagram of the system shown in Fig. 2 which can be used to reconstruct the signal $x_a(t)$ from its sampled version $x[n]$. Where $x_a(t)$ is given by,

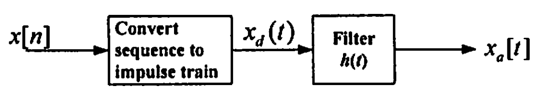


Figure 2: Block diagram to reconstruct $x_a(t)$ from $x[n]$ in Problem 6

$$x_a(t) = \sum_{n=-\infty}^{\infty} x[n]p(t - nT)$$

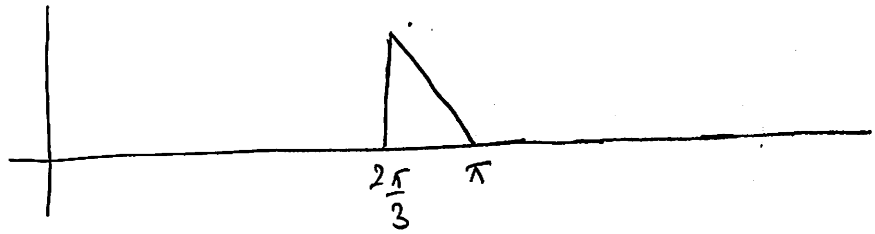
Using the result of Part (b), sketch the frequency response of a filter that can be used to reconstruct $x(t)$.

(d) Derive the filter $h(t)$ which can recover $x_a(t)$ from $x[n]$.

a) $\Omega_1 = 2\pi \times 4000$ rad/s $\qquad \Omega_2 = 2 \times \pi \times 6000$ rad/s

Nyquist Rate $T < \frac{1}{12000}$

The Spectrum is sketched below :



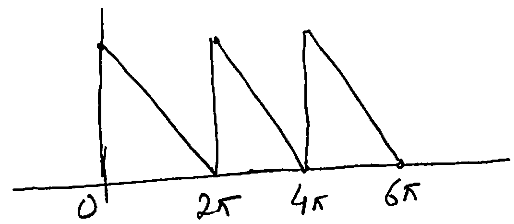
(b) One can sample in a way that the copies of the spectrum do not overlap.

(b) One can sample in a way that the replications in frequency, which occur due to sampling do not overlap with the original.

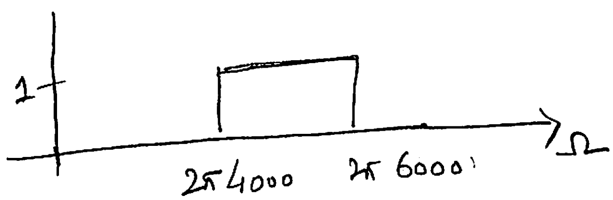
Hence $T < \frac{2\pi}{\Omega_2 - \Omega_1} = \frac{2\pi}{2\pi(6000-4000)}$

$$\Rightarrow T < \frac{1}{2000}$$

The spectrum is sketched below,



(c) We can use a band pass filter to reconstruct the signal:



(d)

$$h(t) = \int_{8000\pi}^{12000\pi} e^{-j\Omega t}\, dt$$

$$= \left. \frac{e^{-j\Omega t}}{-jt} \right|_{8000\pi}^{12000\pi}$$

$$= \frac{-1}{jt}\left[e^{-j\,12000\pi t} - e^{-j\,8000\pi t} \right]$$

$$= \frac{-1}{jt}\left[e^{-j\,8000\pi t} - e^{-j12000t} \right]$$

$$\Rightarrow \quad h(t) = \frac{1}{jt}\left[e^{-j10000t}\left(e^{j2000t} - e^{-j2000t} \right)\right]$$

$$= 2e^{-j10000t} \cdot \frac{\sin 2000t}{t}$$

Problem 7 (10 Points)

Consider an LSI system with impulse response

$$h[n] = \frac{sin\left(\frac{\pi n}{2}\right)}{\pi n}$$

Compute the output signal $y[n]$ for the following input signal,

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-6k]$$

Hint :

$$\frac{1}{\alpha}\sum_{n=-\infty}^{\infty} e^{jn\frac{2\pi t}{\alpha}} = \sum_{n=-\infty}^{\infty} \delta_a(t-n\alpha)$$

$$H(\omega) = \begin{cases} 1 & |\omega| < \pi/2 \\ 0 & \text{else} \end{cases}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} \delta[n-6k] \times e^{-j\omega n k}$$

$$\Rightarrow \quad X(\omega) = \sum_{k=-\infty}^{\infty} e^{-j\omega 6k} \qquad — ①$$

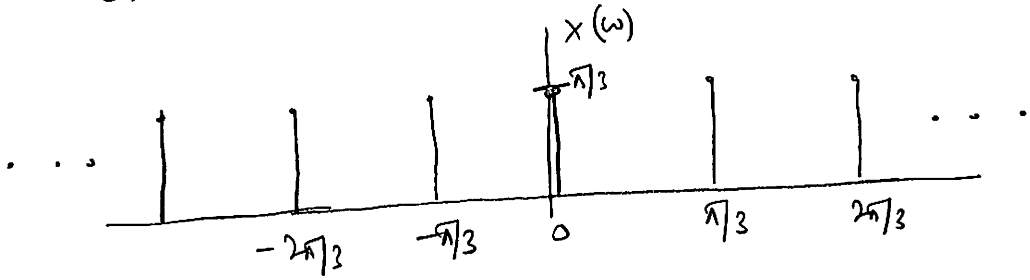
use the hint now,

$$\frac{1}{\alpha}\sum_{n=-\infty}^{\infty} e^{jn\frac{2\pi t}{\tau}} = \delta_a(t-n\alpha)$$

here $6 = \frac{2\pi}{\alpha} \Rightarrow \alpha = \frac{\pi}{3}$

$$\Rightarrow \quad X(\omega) = \frac{\pi}{3}\sum_{k=-\infty}^{\infty} \delta_a\left[\omega - k\frac{\pi}{3}\right]$$

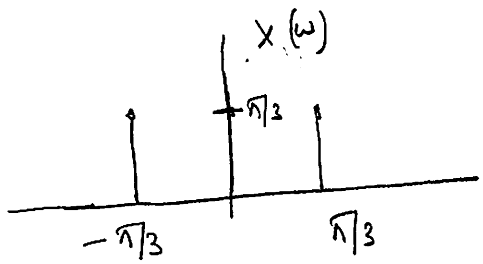
$X(\omega)$ is sketched below,



Note $H(\omega) = 1$ for $|\omega| < \pi/2$

Hence we have,

$Y(\omega) =$



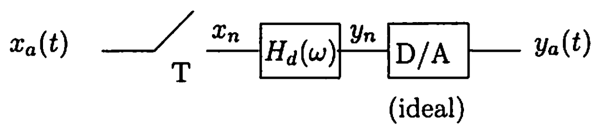
$$\Rightarrow \quad Y(\omega) = \frac{\pi}{3}\left[\delta_a(\omega - \pi/3) + \delta_a(\omega) + \delta_a(\omega + \pi/3)\right]$$

$$y(n) = \frac{1}{6} e^{-j\pi n/3} + \frac{1}{6} + \frac{1}{6} e^{j\pi n/3}$$

$$= \frac{1}{6} + \frac{1}{3}\cos\left(\frac{\pi n}{3}\right)$$

Problem 8 (30 Points)

Consider the system in the following Figure with sampling period T and an ideal D/A converter:



The signal $x_a(t)$ is assumed to be bandlimited to 3500 Hz. Let the signal be sampled at $T = \frac{1}{10000}$s.
Note : In this problem parts (c), (d), and (e) are independent of parts (a) and (b).

(a) It is desired to filter the signal with a bandpass filter that will pass frequencies between 250 Hz to 500 Hz by using the digital filter $H_d(\omega)$, as shown in the Figure above. Sketch the frequency response of the filter, $H_d(\omega)$, for the necessary discrete-time filter, when sampling at rate T. Note: the sketch must clearly show the appropriate frequency values.

(b) Assume that an error occurred in the system such that the lower and the upper cut-off frequencies, ω_l and ω_u of the bandpass filter changed by $\pm\epsilon$. What are the limits on ϵ.

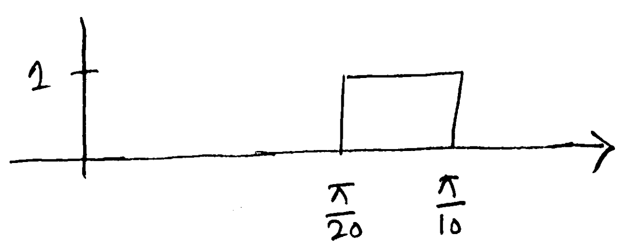
(c) It is now desired to filter the signal using bandstop filter to suppress the frequencies 350 Hz - 700 Hz. What is the maximum sampling period T_s to avoid aliasing.

(d) Sketch the frequency response of the bandstop filter.

(e) Let the sampling period be changed such that $T_{new} = 3 \times T_s$. What is the range of frequencies that will be suppressed by the filter in (d).

(a) $\omega_{min} = \frac{2\pi \times 250}{10000} = \frac{\pi}{20}$

$\omega_{max} = \frac{2\pi \times 500}{10000} = \frac{\pi}{10}$



(b) The Nyquist Rate for this signal is

$T < \frac{1}{7000}$

$\Rightarrow$ based on this sampling rate,

$$\omega_{min} = \frac{2\pi \times 250}{7000} = \frac{\pi}{14}$$

$$\omega_{max} = 2\pi \times \frac{500}{7000} = \frac{\pi}{7}$$

hence ω_{min} cannot be higher than $\frac{\pi}{7}$

ω_{max} cannot be higher than $\frac{\pi}{7}$

$\Rightarrow$ error that can be tolerated

$$= \frac{\pi}{14} - \frac{\pi}{20} = \frac{3\pi}{140}$$

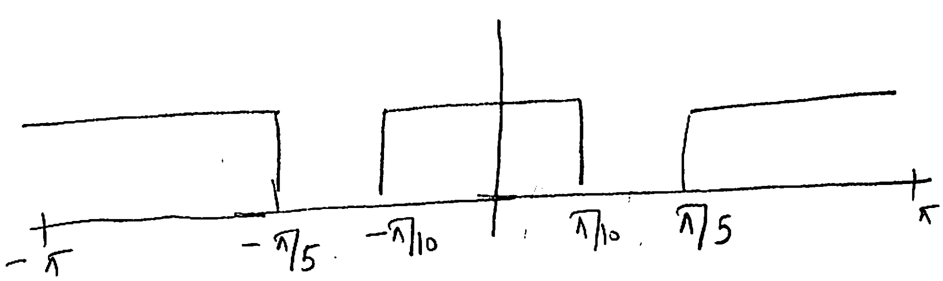
(c) The maximum sampling period is computed in (b)

$$T_s < \frac{1}{7000}$$

(d) The frequencies ω_{min} & ω_{max} are now

$$\omega_{min} = \frac{2\pi \times 350}{7000} = \frac{\pi}{10}$$

$$\omega_{max} = 2\pi \frac{700}{7000} = \frac{\pi}{5}$$



(e) $$\frac{\pi}{10} < 2\pi - 2\pi f \cdot \frac{3}{7000} < \frac{\pi}{5}$$

$$\Rightarrow \quad \frac{9}{5} < \frac{6f}{7000} < \frac{19}{10}$$

$$\Rightarrow \quad f > 2100 \text{ Hz}$$

$$\text{and} \quad f < 2216 \text{ Hz}$$

In addition to the range (350 - 700) Hz the frequencies (2100 - 2216) Hz will also be passed