UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

Department of Electrical and Computer Engineering ECE 410 Digital Signal Processing

Quiz Number 1 Solutions

Thursday, September 6, 2007

Problem 1 (10 points) Given that

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

plot the following discrete-time signal. Label the horizontal and vertical axes for full credit.

$$x[n] = nu[-n+6]u[n-2]$$

We have

$$u[-n+6] = \begin{cases} 1 & (-n+6) \geq 0 \iff n \leq 6 \\ 0 & \text{elsewhere} \end{cases}$$

and

$$u[n-2] = \begin{cases} 1 & (n-2) \geq 0 \iff n \geq 2 \\ 0 & \text{elsewhere} \end{cases}$$

Thus,

$$\boxed{x[n] = \begin{cases} n & 2 \leq n \leq 6 \\ 0 & \text{elsewhere} \end{cases}}$$

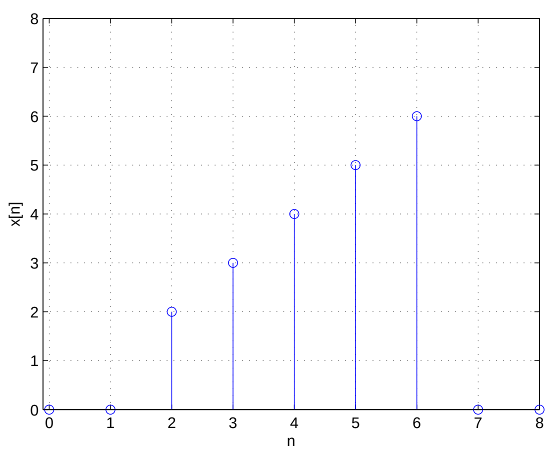

Figure 1: $x[n] = nu[-n+6]u[n-2]$

Problem 2

(a) (15 points)
Determine the discrete-time Fourier Transform (DTFT) $X_d(\omega)$ of the following sequence.

$$x[n] = \begin{cases} (-1)^n & 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Express $X_d(\omega)$ as $X_d(\omega) = R(\omega)e^{j\phi(\omega)}$, where $R(\omega)$ is a purely real function of ω.

(b) (15 points) Evaluate $X_d(\omega)$ at $\omega = \frac{\pi}{2}$, $\omega = \frac{2\pi}{3}$, $\omega = \pi$. Express your solutions in magnitude/phase form (i.e., $Ae^{j\phi}$, where A is a positive real number).

(a)

$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = e^{-j\omega(0)} - e^{-j\omega(1)} + e^{-j\omega(2)} - e^{-j\omega(3)} + e^{-j\omega(4)} = e^{-j2\omega}(e^{j2\omega} - e^{j\omega} + 1 - e^{-j\omega} + e^{-j2\omega})$$

$$\Longrightarrow \boxed{X_d(\omega) = [2\cos(2\omega) - 2\cos(\omega) + 1]e^{-j2w}}$$

(b) i)

$$X_d\left(\frac{\pi}{2}\right) = \left[2\cos(\pi) - 2\cos\left(\frac{\pi}{2}\right) + 1\right] e^{-j\pi} = (-2 - 0 + 1)(-1) \Longrightarrow \boxed{X_d\left(\frac{\pi}{2}\right) = 1e^{j0}}$$

ii)

$$X_d\left(\frac{2\pi}{3}\right) = \left[2\cos\left(\frac{4\pi}{3}\right) - 2\cos\left(\frac{2\pi}{3}\right) + 1\right] e^{j\frac{-4\pi}{3}} = \left[2\left(\frac{-1}{2}\right) - 2\left(\frac{-1}{2}\right) + 1\right] e^{j\frac{2\pi}{3}}$$

$$\Longrightarrow \boxed{X_d\left(\frac{2\pi}{3}\right) = 1e^{j\frac{2\pi}{3}}}$$

iii)

$$X_d(\pi) = [2\cos(2\pi) - 2\cos(\pi) + 1]\, e^{-j2\pi} = (2 - (-2) + 1)(1) \Longrightarrow \boxed{X_d(\pi) = 5e^{j0}}$$

Problem 3 (20 points) Given that the DTFT for a *real* discrete-time signal $x[n]$ is:

$$X_d[\omega] = \begin{cases} 6[\cos(-\frac{2\pi}{3}) + j\sin(\frac{2\pi}{3})] & \text{for } \omega = -\frac{\pi}{7} \\ 12[\cos(\frac{\pi}{4}) - j\sin(-\frac{\pi}{4})] & \text{for } \omega = \frac{3\pi}{7} \\ ???? & \text{elsewhere on } [-\pi, \pi] \end{cases}$$

find $X_d(\frac{\pi}{7})$ and $X_d(-\frac{3\pi}{7})$. Give your solution(s) both in rectangular and polar form.

Note that:

$$X_d\left(-\frac{\pi}{7}\right) = 6\left[\cos\left(-\frac{2\pi}{3}\right) + j\sin\left(\frac{2\pi}{3}\right)\right] = 6\left[\cos\left(\frac{2\pi}{3}\right) + j\sin\left(\frac{2\pi}{3}\right)\right] = 6e^{-j\frac{2\pi}{3}} = -3 + j3\sqrt{3}$$

$$X_d\left(\frac{3\pi}{7}\right) = 12\left[\cos\left(\frac{\pi}{4}\right) - j\sin\left(-\frac{\pi}{4}\right)\right] = 12\left[\cos\left(\frac{\pi}{4}\right) + j\sin\left(\frac{\pi}{4}\right)\right] = 12e^{j\frac{\pi}{4}} = 6\sqrt{2} + j6\sqrt{2}$$

Thus, by the symmetry properties of the DTFT for real-valued signals, we have:

$$X_d\left(\frac{\pi}{7}\right) = X_d^*\left(-\frac{\pi}{7}\right) \Longrightarrow \boxed{X_d\left(\frac{\pi}{7}\right) = -3 - j3\sqrt{3} = 6e^{j\frac{2\pi}{3}}}$$

$$X_d\left(-\frac{3\pi}{7}\right) = X_d^*\left(\frac{3\pi}{7}\right) \Longrightarrow \boxed{X_d\left(-\frac{3\pi}{7}\right) = 6\sqrt{2} - j6\sqrt{2} = 12e^{-j\frac{\pi}{4}}}$$

Problem 4 (20 points) Given that the DTFT of a certain signal $x[n]$ is:

$$X_d(\omega) = \begin{cases} 2 & \text{for } |\omega| \leq \frac{\pi}{2} \\ 0 & \text{elsewhere on } [-\pi, \pi] \end{cases}$$

find the DTFT of $y[n] = x[n]\cos(\frac{\pi}{2}n)$ on $[-\pi, \pi]$. Determine a simplified expression for $y[n]$.

By the modulation property of the DTFT,

$$y[n] = x[n]\cos(\frac{\pi}{2}n) \longleftrightarrow \frac{1}{2}\left[X_d\left(\omega - \frac{\pi}{2}\right) + X_d\left(\omega + \frac{\pi}{2}\right)\right] = Y_d(\omega)$$

Considering that

$$\left|\omega - \frac{\pi}{2}\right| \leq \frac{\pi}{2} \iff -\frac{\pi}{2} \leq \left(\omega - \frac{\pi}{2}\right) \leq \frac{\pi}{2} \iff 0 \leq \omega \leq \pi$$

and

$$\left|\omega + \frac{\pi}{2}\right| \leq \frac{\pi}{2} \iff -\frac{\pi}{2} \leq \left(\omega + \frac{\pi}{2}\right) \leq \frac{\pi}{2} \iff -\pi \leq \omega \leq 0,$$

we have

$$X_d\left(\omega - \frac{\pi}{2}\right) = \begin{cases} 2 & \text{for } 0 \leq \omega \leq \pi \\ 0 & \text{elsewhere on } [-\pi, \pi] \end{cases}$$

and

$$X_d\left(\omega + \frac{\pi}{2}\right) = \begin{cases} 2 & \text{for } -\pi \leq \omega \leq 0 \\ 0 & \text{elsewhere on } [-\pi, \pi] \end{cases}$$

Thus,

$$\boxed{Y_d(\omega) = 1 \text{ on } [-\pi, \pi]}$$

Finally, since the DTFT of a signal is unique and $\sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = 1$, it follows that $\boxed{y[n] = \delta[n]}$.

Problem 5 (20 points) Simplify the following expression

$$\frac{(1-j)^8}{8(\sqrt{3}-j)} + \frac{e^{j\frac{\pi}{6}}}{e^{j\pi}} - (1+j)$$

Give your solution both in rectangular and polar form.

Note that

$$(1-j)^8 = (\sqrt{2}e^{-j\pi/4})^8 = 16e^{-j2\pi} = 16e^{j0}$$

$$8(\sqrt{3}-j) = 16(\frac{\sqrt{3}}{2} - \frac{j}{2}) = 16e^{-j\frac{\pi}{6}}$$

Thus,

$$\frac{(1-j)^8}{8(\sqrt{3}-j)} + \frac{e^{j\frac{\pi}{6}}}{e^{j\pi}} = \frac{16e^{j0}}{16e^{-j\frac{\pi}{6}}} + \frac{e^{j\frac{\pi}{6}}}{-1} = e^{j\frac{\pi}{6}} - e^{j\frac{\pi}{6}} = 0$$

So

$$\frac{(1-j)^8}{8(\sqrt{3}-j)} + \frac{e^{j\frac{\pi}{6}}}{e^{j\pi}} - (1+j) = 0 - (1+j) = \boxed{-1-j = \sqrt{2}e^{-j\frac{3\pi}{4}}}$$