University of Illinois

ECE 410

Spring 2009

Profs. Ahuja & Liang

Midterm Exam I

Thursday, February 26, 2009

Name KEY

Section: 9:00 AM 2:00 PM

Score ____________

Problem	Pts.	Score
1	10	
2	16	
3	6	
4	12	
5	6	
6	13	
7	4	
8	18	
9	15	
Total	100	

Please do not turn this page over until told to do so.

You may not use any books, calculators, or notes other than two sides of a 8.5" x 11" sheet of paper.

GOOD LUCK!

$$\int_{-\infty}^{+\infty} e^{-a|t|} e^{-j\omega t} dt = \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{t(a-j\omega)} dt + \int_{0}^{\infty} e^{-t(a+j\omega)} dt \quad (1\ \text{point})$$

$\boxed{a=5}$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2} \quad (3\ \text{points})$$

(2 point)

Problem 1{10 Points}

Determine the continuous-time Fourier transform (CTFT) of $x_1(t) = e^{-5|t|}$. Let it be denoted by $X_1(\omega)$. The magnitude of $X_1(\omega)$ is given in Figure 1.

1. Sketch the phase of $X_1(\omega)$ in Figure 2.
2. Let $X_2(\omega)$ denote the CTFT of $x_2(t) = e^{-5|t-3|}$. Sketch the magnitude and phase of the $X_2(\omega)$ in Figures 3 and 4.

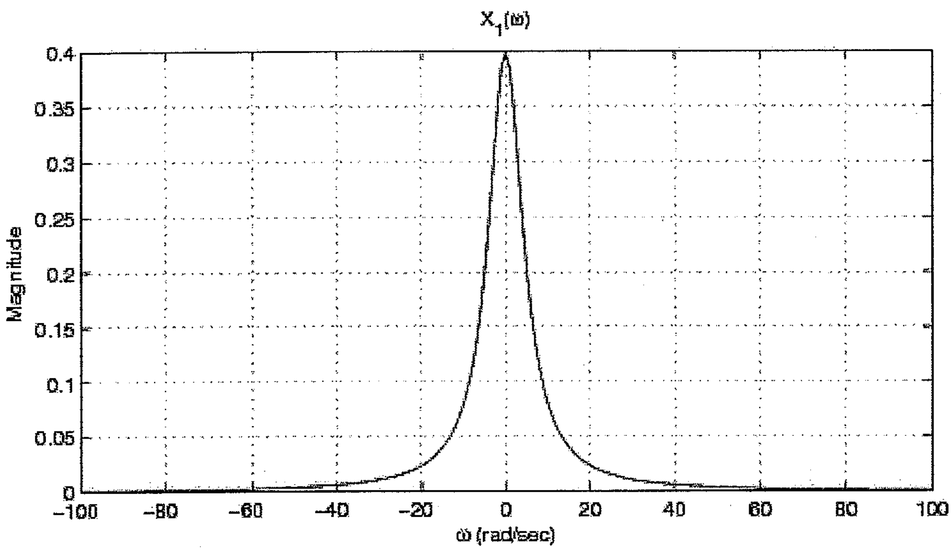
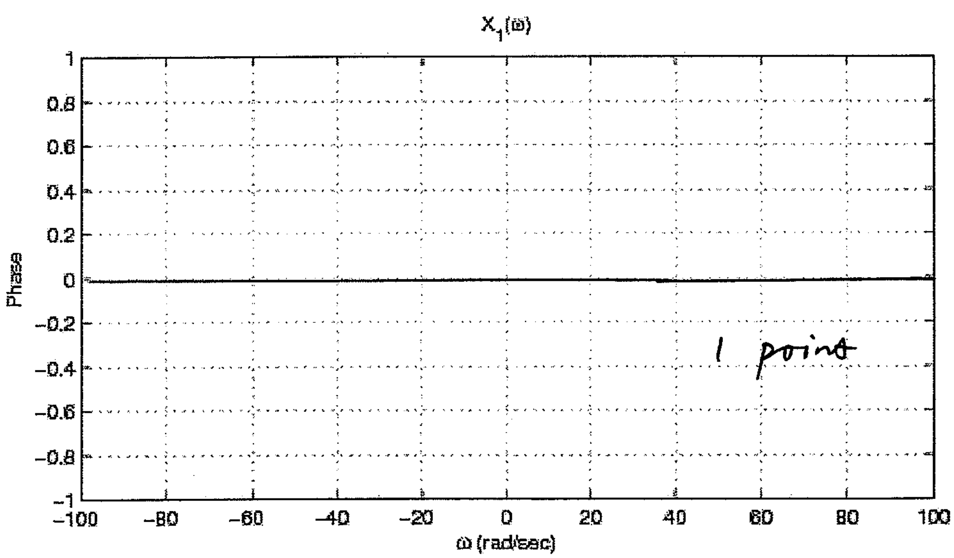


Figure 1: Magnitude of $X_1(\omega)$



phase = 0

1 point

(2 points)

Figure 2: Phase of $X_1(\omega)$

$$\int_{-\infty}^{+\infty} e^{-a|t-3|}\,dt = \int_{-\infty}^{3} e^{a(t-3)} e^{-j\omega t}\,dt + \int_{3}^{\infty} e^{-a(t-3)} e^{-j\omega t}\,dt$$

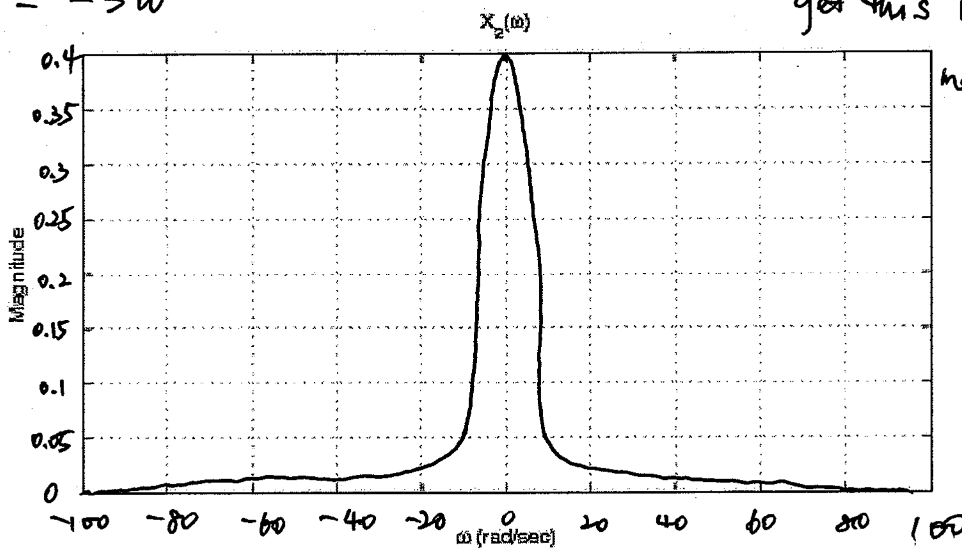
$$= e^{-3a} \cdot \frac{e^{3(a-j\omega)}}{a-j\omega} + e^{3a} \cdot \frac{e^{-3(a+j\omega)}}{a+j\omega}$$

$$= e^{-j3\omega} \cdot \frac{2a}{a^2+\omega^2}$$

$\therefore$ magnitude $= \dfrac{2a}{a^2+\omega^2}$

phase $= -3\omega$

(1 point) If you can finish figure 3 and 4 correctly, you get this 1 point without mathematical derivation

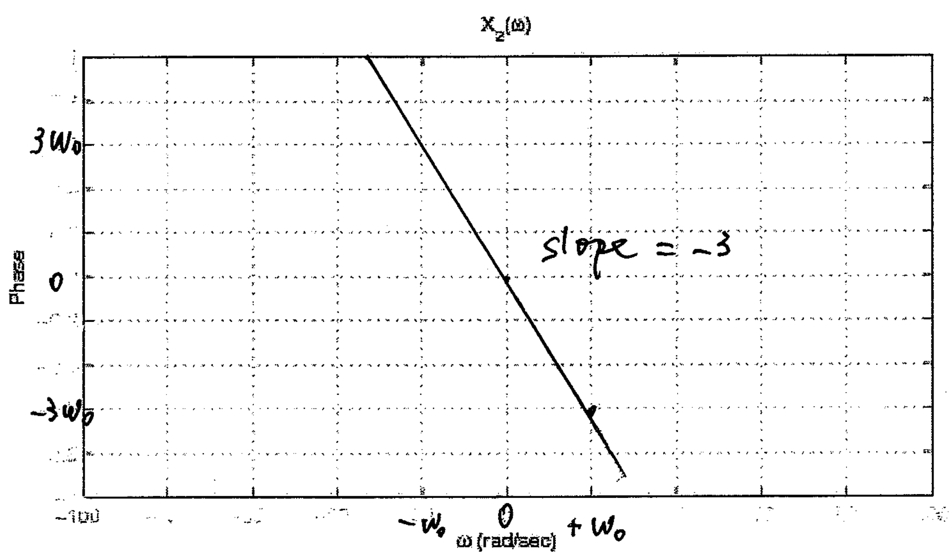


(2 points)

figure 1 point

label 1 point

Figure 3: Magnitude of $X_2(\omega)$



(2 points)

figure 1 point

label 1 point.

W_0 can be any number.

Figure 4: Phase of $X_2(\omega)$

Problem 2{16 Points}

The nonzero elements of a discrete-time sequence $x(n)$ are: $x[-3] = -1$, $x[-1] = 1$, $x[1] = 1$, $x[3] = 1$, $x[4] = 2$, $x[5] = 1$, $x[7] = -1$. For all other n, $x[n] = 0$. Calculate the following WITHOUT obtaining $X_d(\omega)$ first.

1. (a) $X_d(0)$

2 points (1 point)

$$X_d(w) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn} \Rightarrow X_d(0) = \sum_{n=-\infty}^{+\infty} x[n] = 4$$ (1 point)

(b) $\int_{-\pi}^{\pi} X_d(\omega) d\omega$

2 points

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(w) e^{+jwn} dw$$

(1 point) (1 point)

when $n=0 \Rightarrow x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(w) dw \Rightarrow \int_{-\pi}^{\pi} X_d(w) dw = 2\pi x[0] = 0$

(c) $X_d(\pi)$

2 points

$$X_d(\pi) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\pi n} = 1 - 1 - 1 - 1 + 2 - 1 + 1 = 0$$

(1 point) (1 point.)

(d) $\int_{-\pi}^{\pi} |X_d(\omega)|^2 d\omega$

3 points

$$= 2\pi \cdot \left[\sum_{-\infty}^{\infty} |x[n]|^2\right] = 20\pi$$

(2 point) (1 point)

(e) $\int_{-\pi}^{\pi} |\frac{dX_d(\omega)}{d\omega}|^2 d\omega$

3 points

$\because\ n x[n] \Leftrightarrow j \frac{dX_d(w)}{dw}$ (1 point)

(1 point.) (1 point.)

$\therefore \int_{-\pi}^{\pi} \left|\frac{dX_d(w)}{dw}\right|^2 dw = 2\pi \left[\sum_{-\infty}^{\infty} |n x[n]|^2\right] = 316\pi$

2. Let $Y_d(\omega) = Re(X_d(\omega))$. Find the discrete-time sequence $y[n]$ whose DTFT is $Y_d(\omega)$.

4 points

First prove : $x^*[-n] \Leftrightarrow X^*(w)$ (2 point)

$$\sum_{n=-\infty}^{+\infty} x^*[-n] e^{-jwn} = \sum_{k=-\infty}^{+\infty} (x^*[k] e^{jwk}) = \sum_{k=-\infty}^{+\infty} (x[k] e^{-jwk})^* = X^*(w)$$

$$\therefore \frac{x[n] + x^*[-n]}{2} \Leftrightarrow \frac{X_d(w) + X_d^*(w)}{2}$$ (4 points) (1 point)

$$\therefore \text{Let } y[n] = \frac{x[n] + x^*[-n]}{2}$$

$\Rightarrow y[-7] = y[7] = -\frac{1}{2}$ $\quad y[3] = y[-3] = 0$ (1 point.)

$y[-5] = y[5] = \frac{1}{2}$ $\quad y[+1] = y[-1] = 1$

$y[-4] = y[4] = 1$

Problem 3{6 Points}

The continuous-time signal $x_a(t)$ has the continuous-time Fourier transform shown in the figure below. The signal $x_a(t)$ is sampled with sampling interval T to get the discrete-time signal $x[n] = x_a(nT)$.

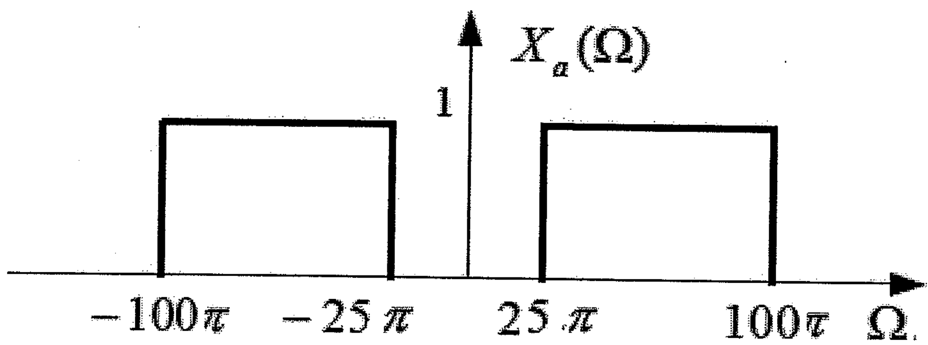
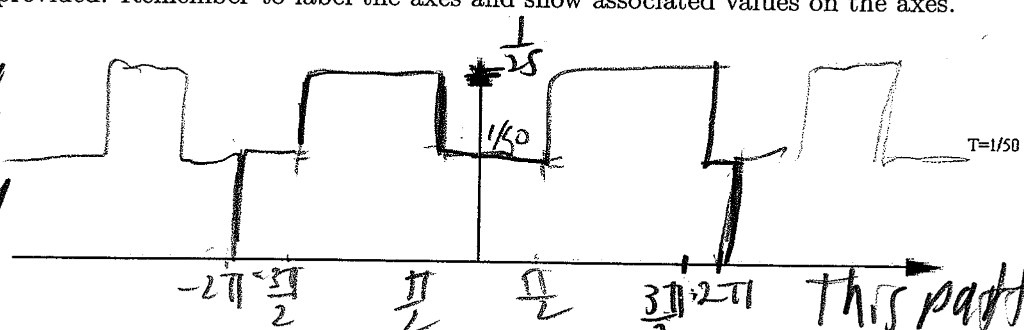


Figure 5: $X_a(\Omega)$

a. Sketch $X_d(\omega)$ (the DTFT of $x[n]$) for the sampling intervals T= 1/200 and 1/50 in the corresponding frames provided. Remember to label the axes and show associated values on the axes.

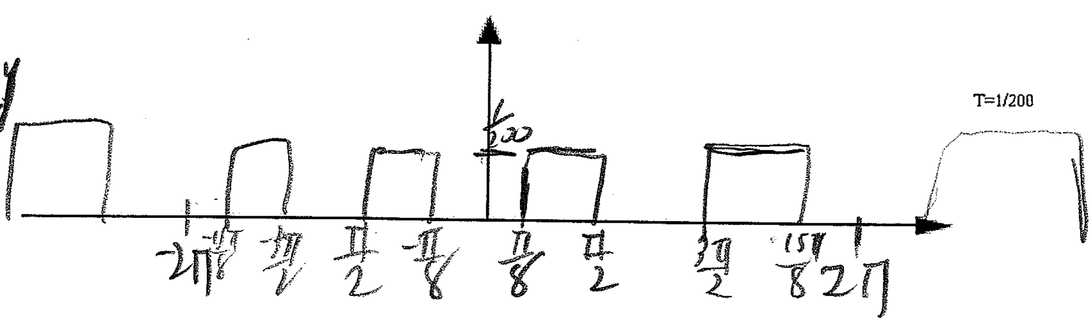
-2 points not showing alaising or -1 not correctly showing resulting figure



this pattern repeats

-1 X axis incorrectly labeled

-1 y axis values



b. What is the minimum sampling rate f_s (Nyquist rate) such that no aliasing will occur in sampling the continuous-time signal?

Nyquist Rate = 100 Hz

-2 if incorrect

$200\frac{rad}{s} \cdot \frac{1}{2\pi} \cdot 2 = 100$ Hz

or -1 for formula error

Problem 4{12 Points}

Compute the discrete-time Fourier transform (DTFT) of the following signals directly using the defining formula.

(a) $x[n] = \sin(\frac{3}{4}n)$

$$\sum_{n=-\infty}^{\infty} \frac{e^{j\frac{3}{4}n} - e^{-j\frac{3}{4}n}}{2j} e^{-jwn} \qquad e^{-jwn} \quad 8\pi\,\delta(w)$$

$$\frac{\pi}{j}\left(\delta(w-a) + \delta(w+a)\right)$$

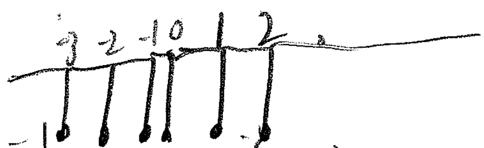
-1 for incorrect indentity

-1 for not using 2π

-1 improper DTFT formula

-1 not showing steps

(b) $x[n] = -u[n+3] + u[n-3]$



$$\sum_{n=-3}^{2} -e^{-jwn}$$

$$\sum_{n=0}^{5} -e^{-jw(n-3)}$$

$$-e^{jw3}\sum_{n=0}^{5} e^{-jw(n)} = -e^{3jw}\frac{(1-e^{-j6w})}{1-e^{-jw}} = \frac{-e^{j3w} + e^{-j3w}}{1-e^{-jw}}$$

-1 incorrect DTFT formula

-1 incorrect range 2 to -3

-2 incorrect summation

(c) $x[n] = (0.4e^{j\pi/2})^n u[n]$

$$\sum_{n=0}^{\infty} (.4e^{j\frac{\pi}{2}})^n e^{-jwn}$$

$$\sum_{n=0}^{\infty} \left(.4e^{j(\frac{\pi}{2}-w)}\right)^n$$

$$\frac{1}{1-.4e^{j(\frac{\pi}{2}-w)}}$$

-1 incorrect DTFT

-3 innocorrect summation

Problem 5{6 Points}

Let $x_a(t) = \sin(7\pi t) + 0.75\cos(5\pi t)$. Let $\{X_m\}_{m=0}^{(M-1)}$ denote the order-M DFT of $x_a(t)$.

Grading

3 pts

1. Given that the analog frequency corresponding to $X[51]$ is 3.984π, determine the relationship between M and T where T is the sampling period.

(-3) No mention of Ω, ω, T, M

(-2) Only stated $\Omega = 3.984\pi$

(-1) Stating $\omega = \frac{2\pi k}{MT}$ without solving

Since 3.984π is positive, no need to adjust ω by 2π.

$$\Omega = \frac{\omega}{T} = \frac{\frac{2\pi k}{M}}{T} = \frac{2\pi k}{MT} \quad \text{where } k = 51 \text{ and } \Omega = 3.984\pi$$

$$MT = \frac{2\pi(51)}{3.984\pi} = \frac{102}{3.984} = 25.6024$$

3pts

2. Given a 2 second long segment of $x_a(t)$, how would you choose the sampling interval T to resolve the sinusoidal components and avoid aliasing? State your criterion for resolvability.

2 second long segment corresponds to $NT = 2$

(1)

Criterion 1	Criterion 2
$NT \geq \frac{4\pi}{\lvert\Omega_1 - \Omega_2\rvert}$	$NT \geq \frac{2\pi}{\lvert\Omega_1 - \Omega_2\rvert}$
$NT \geq \frac{4\pi}{\lvert 7\pi - 5\pi\rvert}$	$NT \geq \frac{2\pi}{\lvert 7\pi - 5\pi\rvert}$
$NT \geq 2$	$NT \geq 1$
$\boxed{2 \geq 2}$ ✓	$\boxed{2 \geq 1}$ ✓

(2) Nyquist: $T < \frac{1}{2f_{max}}$ where $f_{max} = \frac{1}{2\pi}\max(7\pi, 5\pi)$

$f_{max} = 3.5$

$$\boxed{T < \frac{1}{7}}$$

Grading

(-1) For not mentioning Nyquist

(-1) For no resolvability criterion

(-1) For not noticing $NT = 2$

Problem 6{13 Points}

Let $x[n]$ denote the input and $h[n]$ the impulse response of a linear time-invariant system. For the pairs of x and h given in parts (a)-(c), determine the output $y[n]$. You do not need to solve parts (b) and (c) independently; use your knowledge of linearity and time invariance to minimize the work in parts (b) and (c).

Part (a)

5 pts

a. $x[n] = u[n]$ and $h[n] = a^n u[-n-1]$, with $|a| > 1$.

(-4) Just conv. eqn.

(-3) Just convolution sum

(-2) Incorrect evaluation of conv. without separation of cases

(-1) Incorrect evaluation with separation of cases

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} a^k u[-k-1]u[n-k]$$

$$= \begin{cases} \sum_{k=-\infty}^{n} a^k & n \le -1 \\ \sum_{k=-\infty}^{-1} a^k & n > -1 \end{cases} = \begin{cases} \frac{a^{n+1}}{a-1} & n \le -1 \\ \frac{1}{a-1} & n > -1 \end{cases}$$

since $\sum_{i=m}^{n} x^i = \frac{x^{n+1} - x^m}{x-1}$ and $|a| > 1$

4 pts

b. $x[n] = u[n-4]$ and $h[n] = 2^n u[-n-1]$.

Defining $w[n] = u[n] * v[n]$ where $v[n] = 2^n u[-n-1]$

$$w[n] = \begin{cases} 2^{n+1} & n \le -1 \\ 1 & n > -1 \end{cases}$$ from part (a)

Part (b) & (c)

(-3) Just convolution equation

(-2) Incorrectly worked out convolution

(-1) Incorrect application of LSI properties
For each
to answer from (a)

And, $y[n] = u[n-4] * v[n] = w[n-4] = \begin{cases} 2^{n-3} & n \le 3 \\ 1 & n > 3 \end{cases}$

4 pts

c. $x[n] = u[n]$ and $h[n] = (0.5)2^n u[-n]$.

For $v[n]$ and $w[n]$ from part (b), $h[n] = v[n-1]$

$$y[n] = x[n] * h[n] = x[n] * v[n-1] = w[n-1]$$

$$= \begin{cases} 2^n & n \le 0 \\ 1 & n > 0 \end{cases}$$

Problem 7{4 Points}

Give two examples where zero-padding is useful in digital signal processing.

Zero pad to a power of 2 for FFT

Increase resolution in spectural/analysis/DFT

In $e^{-j\frac{\pi}{2}nm}$, j missing → −4
No computation of e term at all → −2
Minor error → −1
Numerical Error → −0

Problem 8{18 Points}

(5 pts) Suppose you are given the 4-point discrete-time sequence $x[n] = \{2, 1, 2, 1\}$ where the first element corresponds to n=0.

a. Compute the DFT $X[m]$ of $x[n]$.

$$X[m] = \sum_{n=0}^{3} x[n]\, e^{-j\frac{\pi}{2}nm}$$

$$\underbrace{\begin{aligned} m=0 &: 2+1+2+1 = 6 \\ m=1 &: 2-j-2+j = 0 \\ m=2 &: 2-1+2-1 = 2 \\ m=3 &: 2+j-2-j = 0 \end{aligned}}_{X[m]} \Rightarrow \{6, 0, 2, 0\}$$

(3 pts) b. Suppose $x[n]$ is the sampled version of continuous-time signal $x(t)$. Using standard notation, give the equation that relates $x(t)$ and $X[m]$

$$X[m] = \begin{cases} \frac{1}{T} X_a\left(\frac{\cancel{2}\pi m}{\cancel{2}4T}\right) & 0 \le m \le 2 \\ \frac{1}{T} X_a\left(\frac{\cancel{2}\pi}{\cancel{2}4T} \langle m-4 \rangle_4\right) & 2 < m \le 3 \end{cases}$$

Only DFT given −2

Any reasonable attempt −2

(5 pts) c. What is the DFT of $X[m]$?

$$\{4\, x_{\langle 4-m \rangle_4}\}_{m=0}^{3}$$ = x[n] scaled by 4 and flipped except for x[0]

$$= 4\{2\ 1\ 2\ 1\} = \{8\ 4\ 8\ 4\}$$

(5 pts) d. Suppose you are given another sequence $w[n] = \{1, 2, 3, 4\}$. Let $C[m]$ denote the cyclic convolution of $x[n]$ and $w[n]$. Compute the values of $C[1]$ and $C[3]$.

Only DFT −4

If X in place of x −3

If only formula no values −2

$$C[m] = \sum_{l=0}^{3} x[l]\, w_{\langle m-l \rangle_4}$$

$$C[1] = \sum_{l=0}^{3} x[l]\, w_{\langle 1-l \rangle_4} = 4+1+8+3 = 16$$

$$C[3] = \sum_{l=0}^{3} x[l]\, w_{\langle 3-l \rangle_4} = 8+3+4+1 = 16$$

Normal convolution instead of cyclic −3

Only formula −3

Understanding of what is causal / time-invariant and linear is worth 1 point even your result is wrong!

Problem 9{15 Points}

In (a)-(c), $x[n]$ denotes the input of a system and $y[n]$ denotes its output.

(a) $y[n] = x[2n]$
Is the system causal? (Yes/No)
Justify your answer:

(5 points)

Let $n=1 \Rightarrow y[1] = x[2]$

Current y depends on future x, so it is non-causal.

(b) $y[n] = n^2 x[2n]$
Is the system time-invariant? (Yes/No)
Justify your answer:

(5 points)

$$x[2(n+n_0)] \rightarrow n^2 x[2(n+n_0)] = y_1[n]$$

$$y[n+n_0] = (n+n_0)^2 x[2(n+n_0)]$$

$\because y[n+n_0] \neq y_1[n] \quad \therefore$ Time-variant.

(c) $y[n] = x^3[2n]$
Is the system linear? (Yes/No)
Justify your answer:

(5 points)

$$a x_1[n] \rightarrow a y_1[n] = a x_1^3[2n]$$

$$b x_2[n] \rightarrow b y_2[n] = b x_2^3[2n]$$

$$x[n] = a x_1[n] + b x_2[n] \rightarrow y[n] = (a x_1[n] + b x_2[n])^3$$

$$\neq a y_1[n] + b y_2[n]$$

$\therefore$ It is NOT linear!