ECE 313: Probability with Engineering Applications

Chapter 1: Foundations

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1.2: Axioms of probability

A random experiment is modeled by a *probability space*, which is a triplet $(\Omega, \mathcal{F}, P)$.

- Ω represents the set of all possible outcomes. For example, for a random experiment that involves rolling a six sided fair die, $\Omega = \{1, 2, 3, 4, 5, 6\}$. Ω is said to be finite if it has finitely many elements.
- An *event* A is a subset of Ω. An event A is said to occur or to be true if the outcome of a random experiment ω is an element of A. For example, for the above die experiment, A can be equal to $\{1, 3, 5\}$, the set of odd numbers.
- $\mathcal{F}$ represents the set of all possible events. You can think of $\mathcal{F}$ as the the set of all subsets of Ω.
- P is a probability measure on $\mathcal{F}$, which assigns a probability $P(A)$, to each event $A \in \mathcal{F}$. For example, for a finite Ω, you can set $P(A) = \frac{|A|}{|\Omega|}$, where $|A|$ represents the number of elements A.

Observe that $A \cup A^c = \Omega$ and $A \cap A^c = \emptyset$. Two events A, B are said to be *mutually exclusive* if $A \cap B = \emptyset$. They are said to be *mutually exhaustive* if $A \cup B = \Omega$. Sets A and B form a partition if they are mutually exclusive and exhaustive. Similarly, we have that

- $A_1, A_2, \cdots$ are mutually exclusive if $A_i \cap A_j = \emptyset$ for all i and j
- $A_1, A_2, \cdots$ are mutually exhaustive if $A_1 \cup A_2 \cup \cdots = \Omega$
- The list of events $A_1, A_2, \cdots$ forms a partition if $A_1, A_2, \cdots$ are mutually exclusive and exhaustive

De Morgan's law:

- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$

Event axioms:

1. $\Omega \in \mathcal{F}$
2. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
3. If $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$. More generally, if $A_1, A_2, \cdots$ are all in $\mathcal{F}$, then $A_1 \cup A_2 \cup \cdots$ is in $\mathcal{F}$.

Event properties: If the above axioms are satisfied, then $\mathcal{F}$ has the following properties.

- $\emptyset \in \mathcal{F}$
- If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$. More generally, if $A_1, A_2, \cdots$ are all in $\mathcal{F}$, then $A_1 \cup A_2 \cup \cdots$ is in $\mathcal{F}$.

Probability axioms:

1. $\forall A \in \mathcal{F}$, $P(A) \geq 0$
2. If $A, B \in \mathcal{F}$ and are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$. More generally, if $A_1, A_2, \cdots$ are all in $\mathcal{F}$ and are mutually exclusive, then $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$.
3. $P(\Omega) = 1$

Probability measure properties: If the above axioms are satisfied, then P has the following properties.

- $\forall A \in \mathcal{F},\ P(A^c) = 1 - P(A)$
- $\forall A \in \mathcal{F},\ P(A) \leq 1$
- $P(\emptyset) = 0$
- If $A \subseteq B$, then $P(A) \leq P(B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

1.3: Calculating the size of various sets

Principle of counting: If there are m ways to select one variable and n ways to select another variable, and if these two selections can be made independently, then there is a total o mn ways to make the pair of selections. For example, there are 2^8 possible 8-bit binary strings.

Orderings and permutations: The number of ways to order n distinct objects is $n! = n \cdot (n-1) \cdots 2 \cdot 1$. For example, there are 4! orderings of the letters A, B, C, and D. An ordering of n distinct objects is called a *permutation*.

Principle of over counting: If an object appears k times in a list of n objects and if the other $n - k$ objects are all distinct, then we can order the n objects in $n!/k!$ **distinct ways**. More generally, if objects $1, 2, ..., l$ appear $k_1, k_2, ..., k_l$ times, respectively, and if the other $n - k_1 \cdots - k_l$ objects are all distinct, then we can order the n objects in

$$\frac{n!}{k_1! \cdot k_2! \cdots k_l!}$$

distinct ways. For example, there are $6!/(3! \cdot 2!)$ distinct orderings of the letters ILLINI.

Choosing k unordered objects from a set of n distinct objects: The number of subsets of size k of a set of n distinct objects is given by

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

Notice that the order of the k elements doesn't matter here because sets are "unordered". Also, observe that $\binom{n}{k} = \binom{n}{n-k}$ because instead of choosing k objects, we can choose $n - k$ objects and consider the other k objects. For example, there are $\binom{9}{5}$ ways to choose 5 out of 9 basketball players.

1.4: Probability experiments with equally likely outcomes

Please read examples 1.4.1, 1.4.2, and 1.4.3 carefully.

1.5: Sample spaces with infinite cardinality

If a random experiment can generate infinitely many outcomes, then $|\Omega| = \infty$. We distinguish between two important types of sample spaces with infinite cardinality.

- If $\Omega = \{\omega_1, \omega_2, \cdots\}$, then Ω is countably infinite (i.e., we can list the elements Ω sequentially without skipping any intermediate elements). For example, if $\Omega = \mathbb{N}$ (the set of natural numbers), $\Omega = \mathbb{Z}$ (the set of all integers), or $\Omega = \mathbb{Q}$ (the set of all rational numbers), then Ω is countably infinite.
- If $\Omega = \{\omega : 0 \leq \omega \leq 1\}$ or $\Omega = \mathbb{R}$, then Ω is uncountably infinite. Equivalently, we say that Ω is not countable.

The concept of countable and uncountable sample spaces applies not only to probability sample spaces, but also for arbitrary spaces/sets.